University of Szczecin

Institute of Mathematics

Real hyperfields

Doctoral Thesis

Author:
Hanna Weronika
Stojałowska

Supervisor:
prof. Franz-Viktor
Kuhlmann

Szczecin, 2024

To Katarzyna Zakrzewska
in gratitude for showing me
the beauty of mathematics

Acknowledgements

The author would like to thank certain people who participated in this work. First and foremost, I would like to thank my supervisors Franz-Viktor Kuhlmann and Katarzyna Kuhlmann for their invaluable help, hours of discussions, most valuable comments, and for every meeting, conversation and mountain trip. In addition, huge thanks are due to Alessandro Linzi and Dawid Kędzierski, co-authors of the articles, for fruitful collaboration, which I hope to continue in the future. Next, I would like to thank other members of my former research group: Hanna Ćmiel, Piotr Szewczyk and Piotr Błaszkiewicz, for all the hours spent working together and for the unforgettable scientific (and non-scientific) trips.

I would also like to thank my parents, my grandparents and my sister for their love, care and support in every aspect and my friends for every kind word and help. In particular, I would like to thank Edyta Kowalewska for her valuable remarks.

Finally, I would like to thank my husband Mariusz and my daughter Zuzanna for the strength they give me every day. With you, nothing is impossible.

OŚWIADCZENIE DOKTORANTA

Oświadczam, że moja praca pt.: Real hyperfields

została napisana przeze mnie samodzielnie,

a. nie narusza praw autorskich w rozumieniu ustawy z dnia 14 lutego 1994 roku o prawie autorskim i prawach pokrewnych (Dz.U. 2019 r. poz. 1231) oraz dóbr osobistych chronionych prawem,
b. nie zawiera danych i informacji, które uzyskałem / uzyskałam w sposób niedozwolony,
c. nie była podstawą nadania tytułu naukowego lub zawodowego ani mnie ani innej osobie.

Ponadto oświadczam, że treść pracy przedstawionej przeze mnie do obrony, zawarta na przekazanym nośniku elektronicznym jest identyczna z jej wersją drukowaną.

Szczecin, dn. ……. ……

podpis doktoranta

OŚWIADCZENIE

Wyrażam zgodę na udostępnienie mojej pracy doktorskiej pt.: Real hyperfields.

Szczecin, dn. ……. ……

podpis doktoranta

Contents

Introduction

In 1934 the French mathematician F. Marty presented at the 8th Scandinavian Congress of Mathematicians the concept of a generalization of the notion of a group (see [31]). He called this algebraic structure a hypergroup. From this point on, hypercompositional algebra came into being. This is a branch of algebra dealing with algebraic structures in which (at least some) operations are transformed into hyperoperations, i.e., operations that result in a set of elements, not a single element. Marty died during World War II six years later, but his idea survived. Today, hypercompositional algebra is a widely studied branch of mathematics that finds a number of applications (see [9], [10]).

Seven years before Marty introduced hypercompositional algebra, E. Artin and O. Schreier in [2] had published the foundations of the theory of ordered fields. Using this theory Artin, in the same year in [1], gave a positive answer to the question known as Hilbert's 17th Problem: *Suppose that $f \in \mathbb{R}[X_1, \ldots, X_n]$ is a polynomial that takes only non-negative values over the reals. Is f a finite sum of squares of rational functions?* Moreover, the theory of ordered fields gave rise to real algebraic geometry, which is a sub-branch of algebraic geometry (see e.g. [6]).

Hyperfields are a generalization of fields where addition is multivalued and multiplication is single-valued. They were introduced by M. Krasner in [24].

In 2006 in [34] M. Marshall introduced the notion of a real hyperfield. A hyperfield F is said to be real if it admits a positive cone, i.e., a subset P of F such that $P+P \subseteq P$, $P \cdot P \subseteq P$, $P \cap -P = \emptyset$, $P \cup -P = F^*$, where $F^* := F \setminus \{0\}$. Moreover, Marshall introduced the notion of a prepositive cone in a hyperfield. Furthermore, he generalized also some aspects of Artin-Schreier theory to hyperfields. Namely, he proved that in the hyperfield case the following results are valid: every maximal prepositive cone is a positive cone and a prepositive cone is an intersection of all positive cones in which it is contained. Moreover, he proved the analogue of Artin-Schreier's theorem for hyperfields, i.e., that a hyperfield F is real if and only if $-1 \notin \sum(F^*)^2$. We recall these results in Section 2.1. The work of Marshall was later further developed in [16], where M. Marshall and P. Gładki investigated the orderings of higher level in hyperfields. However, because of a lack of valuation-theoretic tools, the authors were not able to study orderings of higher levels in hyperfields in general. In this thesis we address this problem. Our aim is to prove an analogue of the Baer-Krull

theorem for hyperfields and develop the theory of real hyperfields. The compatibility between positive cones and valuations in real hyperfields allows us to generalize several aspects of the theory of real hyperfields. However, some facts from the theory of ordered fields are no longer true for real hyperfields. We point them out.

In the first chapter we recall and systematize the basic concepts related to hypercompositional algebra. We recall the definitions of a hypergroup from [31], of a canonical hypergroup from [38] and of a hyperring and hyperfield from [24]. In [23] Krasner presented a contruction of a hyperfield which is a quotient of a field by a multiplicative subgroup of this field. We recall this construction. We show also Massouros's example from [32] which shows that there exist hyperfields which are not coming from this construction. We recall some results from hyperring theory from [18] and [11]. In particular, we present the construction of a hyperring of fractions, introduced in [11], and we give all the necessary details in the proofs. Moreover, we show an example from [46] of a hyperfield in which double distributive law does not hold. We also present the result from [7] which says that if a hyperfield fulfills the double distributive law, then it is a stringent hyperfield.

The terminology in the literature is inconsistent in some cases. For example, the notion of a quotient hyperfield is usually used for a quotient of a hyperring by a hyperideal as introduced by Jun in [18] (see Proposition 1.3.20). However, the same notion is used also for a hyperfield which comes from the Krasner contruction in [23] (see Theorem 1.2.5). We address the second case by the name of factor hyperfield. Moreover, there are also two ways in the literature to define a subhyperring. Some authors define a subhyperring to be a subset of a hyperring which is additively and multiplicatively closed (see e.g. [11]). Others (see e.g. [18]) define it to be a subset of a hyperring which with induced operations is a hyperring itself. We give the name of strict subhyperring in the first case (Definition 1.3.1).

Among a lot of valuable articles in hypercompositional algebra there are also some in which gaps occur and some of the proofs are not correct. An example is the article [37]. The proofs from that paper have been corrected and the gaps have been filled by the author with co-author A. Linzi in [30]. Having in mind this problem, we give all the necessary details in the proofs and sometimes we give new proofs of the results from the literature in this thesis.

In the second chapter we present Marshall's theory of real hyperfields. We pay special attention to real factor hyperfields. In particular, we show the relation between the positive cones of a field and the positive cones of a real factor hyperfield arising from that field. We give several interesting examples of real hyperfields, both factor and not factor.

In the third chapter we develop the basic valuation theory of hyperfields introduced by B. Davvaz and A. Salasi in [11]. Furthermore, we focus on valued factor hyperfields. We show the relation between the valuations on a field and on the factor hyperfield arising from that field. We introduce the notion of a Prüfer hyperring.

The fourth chapter is the main part of the dissertation and contains only new results. First, we

introduce the notion of a valuation compatible with a positive cone (Proposition 4.1.7), and of the natural valuation (Definition 4.1.8). We prove the Baer-Krull theorem for hyperfields (Theorem 4.1.23).

Having a real field (K, P), we can introduce a strict linear order $<$ associated with P in the following way:

$$a < b :\Leftrightarrow b - a \in P \quad \text{for } a, b \in K.$$

In the case of real hyperfields a positive cone is not necessarily associated with a strict linear order. That was already pointed out by Marshall in [34]. Nevertheless, the relation on a real hyperfield (F, P) given by

$$a < b :\Leftrightarrow b - a \subseteq P \quad \text{for } a, b \in F$$

is a strict partial order (see Section 4.2). In the case of real fields we say that a valuation is compatible with a positive cone P if and only if its valuation ring is convex with respect to the strict linear order associated with P. In the case of hyperfields, the existence of a strict partial order allows us to consider the notion of convexity. However, in real hyperfields only one implication is true: if a valuation is compatible with a positive cone P, then its valuation hyperring is convex with respect to the strict partial order associated with P (Lemma 4.2.39). The reverse implication is not true in general (Proposition 4.2.41).

Moreover, we introduce the notion of a real holomorphy hyperring (Definition 4.3.1) and prove that the real holomorphy hyperring is a Prüfer hyperring (Theorem 4.3.5). We also introduce the notion of a real place on a real hyperfield (Definition 4.4.2) and show that the topological space of real places on a real hyperfield has analogous properties to those in the classical theory of ordered fields (Theorem 4.4.5).

In Section 4.2.1 we pay special attention to real factor hyperfields. We study the strict partial order relation associated with their positive cones and focus on the elements which are not comparable. We show that every real factor hyperfield is a union of a chain of antichains and determine the number of elements in each antichain (Proposition 4.2.36).

Some results of this thesis have already been published in [26], a paper written by the author with co-authors K. Kuhlmann and A. Linzi.

Chapter 1

Hypercompositional algebraic structures

In this chapter we introduce variants of definitions of several algebraic structures: hypergroup, canonical hypergroup, hyperring and hyperfield. We provide the basic properties and give examples of these structures. In Section 1.3 we focus on the theory of hyperrings and recall the constructions which have appeared already in the literature. We provide the details of some proofs.

1.1 Hypergroups

The idea of hypercompositional algebraic structures is based on the concept of generalizing the definition of a given algebraic structure by converting a single-valued operation into a multi-valued operation. We begin by focusing on the implementation of this idea to the definition of a group, which was also chronologically done first. To this end, let us begin by recalling the classical definition of a group.

Definition 1.1.1. Let G be a nonempty set with an operation $+ : G \times G \to G$ and let 0 be an element in G. Then $(G, +, 0)$ is called a *group* if

(G1) the operation $+$ is associative, i.e., $(x + y) + z = x + (y + z)$ for every $x, y, z \in G$,

(G2) $x + 0 = 0 + x = x$ for every $x \in G$,

(G3) for each element $x \in G$ there exists an element $y \in G$ such that $x + y = y + x = 0$.

However, we can define a group also in the following, alternate way.

Definition 1.1.2. Let G be a nonempty set with an operation $+ : G \times G \to G$. Then $(G, +)$ is called a *group* if

(g1) the operation $+$ is associative, i.e., $(x + y) + z = x + (y + z)$ for every $x, y, z \in G$,

(g2) the operation $+$ is reproductive, i.e., $x + G = G + x = G$ for every $x \in G$, where

$$G + x = \{g + x \mid g \in G\} \quad \text{and} \quad x + G = \{x + g \mid g \in G\}.$$

Proposition 1.1.3 (Theorem 2, Theorem 3, [33])**.** *Definition 1.1.1 and Definition 1.1.2 are equivalent.*

Proof. Take a nonempty set G with an operation $+$ and $0 \in G$ and assume that $(G, +, 0)$ satisfies axioms (G1) - (G3). We need to show that it satisfies axiom (g2). Clearly $x + G \subseteq G$ for every $x \in G$. To prove the converse inclusion, take any $a \in G$. Then

$$a = 0 + a = (x - x) + a = x + (-x + a) \in x + G.$$

Hence $G = x + G$. Similarly we prove that $G = G + x$ for every $x \in G$. So we see that Definition 1.1.1 implies Definition 1.1.2.

Take now a nonempty set G with an operation $+$ and assume that $(G, +)$ satisfies axioms (g1) and (g2). We need to show that it satisfies axioms (G2) and (G3). Take $x \in G$. From axiom (g2) we know that there exists some element $0 \in G$ such that $x + 0 = x$. Take now an element $y \in G$. Again, from (g2), there exists $z \in G$ such that $z + x = y$. Hence, using (g1), we have

$$y + 0 = (z + x) + 0 = z + (x + 0) = z + x = y.$$

So $y + 0 = y$ for every $y \in G$. In an analogous way, we show that there exists $\tilde{0} \in G$ such that $\tilde{0} + y = y$ for every $y \in G$. Finally,

$$\tilde{0} = \tilde{0} + 0 = 0,$$

which shows that axiom (G2) is fulfilled.

Take $x \in G$. From (g2) we know that there exist $y, t \in G$ such that $x + y = 0$ and $t + x = 0$. Moreover, using axiom (g1), we have

$$y = 0 + y = (t + x) + y = t + (x + y) = t + 0 = t.$$

This shows that axiom (G3) is fulfilled. □

Definition 1.1.4. Let G be a nonempty set and $P^*(G)$ the family of nonempty subsets of G. A *hyperoperation* is a function

$$+ : G \times G \to P^*(G)$$

which to every pair (x, y) associates a nonempty subset of G denoted by $x + y$.

For a subset $A \subseteq G$ and $x \in G$ we set

$$A + x := \bigcup_{a \in A} a + x \quad \text{and} \quad x + A := \bigcup_{a \in A} x + a.$$

The following definition of a hypergroup was introduced by F. Marty in [31].

Definition 1.1.5 ([31])**.** Let G be a nonempty set with a hyperoperation $+ : G \times G \to P^*(G)$. Then $(G, +)$ is called a *hypergroup* if

(H1) the hyperoperation $+$ is associative, i.e., $(x + y) + z = x + (y + z)$ for every $x, y, z \in G$,

(H2) the hyperoperation $+$ is reproductive, i.e., $x + G = G + x = G$ for every $x \in G$.

Lemma 1.1.6. *Let $(G, +)$ be a hypergroup and let $H_1, H_2 \subseteq G$ be subsets of G such that $H_1 \subseteq H_2$. Then for any $a \in G$,*

$$a + H_1 \subseteq a + H_2.$$

Proof. Take $a \in G$ and choose $x \in a + H_1$. Then there exists $h_1 \in H_1$ such that $x \in a + h_1$. Since $H_1 \subseteq H_2$, $h_1 \in H_2$. Hence, $x \in a + h_1 \subseteq a + H_2$. This shows that $a + H_1 \subseteq a + H_2$. □

Comparing Definition 1.1.2 and Definition 1.1.5 we see that the notion of hypergroup is a generalization of the notion of a group. We consider the following special class of hypergroups, which will be of interest for us:

Definition 1.1.7 (Definition 1.1, [38])**.** A *canonical hypergroup* is a tuple $(H, +, 0)$, where $+ : H \times H \to P^*(H)$ is a hyperoperation and 0 is an element of H such that the following axioms hold:

(C1) the hyperoperation $+$ is associative, i.e., $(x + y) + z = x + (y + z)$ for all $x, y, z \in H$,

(C2) $x + y = y + x$ for all $x, y \in H$,

(C3) for every $x \in H$ there exists a unique $-x \in H$ such that $0 \in x + (-x)$ (the element $-x$ is called an inverse of x),

(C4) $z \in x + y$ implies $y \in -x + z$ for all $x, y, z \in H$.

The axiom (C4) is called the *reversibility axiom*.

Remark 1.1.8 ([32]). From the definition of a canonical hypergroup we can conclude the following property:

$$x + 0 = \{x\} \text{ for every } x \in H.$$

Indeed, take $y \in x + 0$. Then $0 \in -x + y$ from axiom (C4). Now from axiom (C3) we have that $x = y$, because the inverse is unique.

Remark 1.1.9. The zero element is unique in a canonical hypergroup. Indeed, assume that 0 and $0'$ are zeros in the canonical hypergroup H. Then

$$\{0\} = 0 + 0' = \{0'\},$$

where we used twice Remark 1.1.8 and axiom (C2). Hence, $0 = 0'$.

Note that we can think about the canonical hypergroup as a generalization of the notion of an abelian group in the sense of Definition 1.1.1.

Remark 1.1.10. A canonical hypergroup is a hypergroup in the sense of Definition 1.1.5. Indeed, the inclusions $x + H \subseteq H$ and $H + x \subseteq H$ for every $x \in H$ are clear. To see the reverse inclusions, take any $y \in H$. Then

$$y \in y + 0 \subseteq y + (x + (-x)) = (y + (-x)) + x,$$

so there exists $z \in (y + (-x)) \subseteq H$ such that $y \in z + x \subseteq H + x$. We obtain that $H \subseteq H + x$, and using axiom (C2), also that $H \subseteq x + H$. Hence, $H + x = H$ and $x + H = H$.

However, in the next example we will see that there exist hypergroups which are not canonical hypergroups.

Example 1.1.11.

1. Every abelian group G can be turned into a canonical hypergroup by identifying an element $x + y \in G$ with a singleton $\{x + y\}$.

2. Define the following hyperoperation $+$ on $\mathbb{R}$:

$$x + y := \begin{cases} (\min\{x, y\}, \max\{x, y\}) & \text{if } x \neq y \\ \{x\} & \text{if } x = y. \end{cases},$$

where $(\min\{x, y\}, \max\{x, y\})$ is an open interval. Then $H := (\mathbb{R}, +)$ is a hypergroup in the sense of Definition 1.1.5. Indeed, to prove that H fulfills axiom (H1), we will consider the following cases:

- if $x = y = z$, then $(x + y) + z = (x + x) + x = \{x\} + x = \{x\} = x + \{x\} = x + (x + x) = x + (y + z)$,
- if $x = y \neq z$, then $(x + y) + z = (x + x) + z = \{x\} + z = (\min\{x, z\}, \max\{x, z\}) = x + (\min\{x, z\}, \max\{x, z\}) = x + (x + z) = x + (y + z)$,
- if $x \neq y = z$, then $(x + y) + z = (x + y) + y = (\min\{x, y\}, \max\{x, y\}) + y = (\min\{x, y\}, \max\{x, y\}) = x + y = x + \{y\} = x + (y + y) = x + (y + z)$,
- if $x = z \neq y$, then $(x + y) + z = (x + y) + x = (\min\{x, y\}, \max\{x, y\}) + x = (\min\{x, y\}, \max\{x, y\}) = x + (\min\{x, y\}, \max\{x, y\}) = x + (y + x) = x + (y + z)$,
- if $x \neq y \neq z$, then $(x+y)+z = (\min\{x, y\}, \max\{x, y\})+z = (\min\{x, y, z\}, \max\{x, y, z\}) = x + (\min\{y, z\}, \max\{y, z\}) = x + (y + z)$.

Hence, axiom (H1) is fulfilled. To show that (H2) is valid, observe first that $x + H \subseteq H$ and $H + x \subseteq H$ for every $x \in H$. To prove reverse inclusions, choose $x \in H$. Take $y_1 \in H$. Then $y_1 \in x + y_2 \subseteq x + H$, where we choose $y_2 \in H$ in the following way:

$$\begin{cases} y_2 > y_1 \text{ if } x < y_1, \\ y_2 = y_1 \text{ if } x = y_1, \\ y_2 < y_1 \text{ if } x > y_1. \end{cases}$$

Hence, $H \subseteq x + H$ for every $x \in H$. Similarly we can show that $H \subseteq H + x$ for every $x \in H$. We have proven that $(\mathbb{R}, +)$ is a hypergroup. However, it is not a canonical hypergroup, since there is no unique identity element. Indeed, suppose that $y \in H$ is the identity element and take any $x \in H$. Then $x + y = \{x\}$ if and only if $y = x$, so the identity is not unique.

3. (A variant of Example 3.4 in [46]). Consider the unit interval $[0, 1] \subseteq \mathbb{R}$ with a hyperoperation defined as follows:

$$x + y := \begin{cases} \max\{x, y\} & \text{if } x \neq y, \\ [0, x] & \text{if } x = y. \end{cases}$$

Then $H := ([0, 1], +, 0)$ is a canonical hypergroup. Indeed, axiom (C2) clearly holds. Moreover, for any $x \in H$ the unique inverse element is $x \in H$, so axiom (C3) is fulfilled. Assume that $z \in x + y$. If $x \neq y$ then, without loss of generality, let $x < y$. Hence, $z \in x + y = \{y\}$. Then $\{y\} = x + y = -x + z$. If $x = y$, then $z \in x + y = x + x = [0, x]$, i.e., $z \leq x$. Then $x \in x + z = -x + z$. So axiom (C4) holds in both cases. To see that the hyperoperation $+$ is associative, take $x, y, z \in H$. We will consider the following cases:

- if $x = y = z$, then $(x + y) + z = [0, x] = x + (y + z)$,
- if there exists the largest element among x, y, z, we can assume that $x > y$ and $x > z$. Then
$$(x + y) + z = \{x\} = x + (y + z),$$
- if two elements among x, y, z are equal and bigger than the third one, we can assume that $x = y > z$. Then
$$(x + y) + z = [0, x] + z = [0, x] = x + y = x + (y + z).$$

This proves that axiom (C1) is fulfilled. We have proven that H is a canonical hypergroup.

1.2 Hyperrings and hyperfields

Hyperrings and hyperfields were introduced in 1956 by M. Krasner in [24] in connection with his work on valued fields. A hyperring is a hypercompositional algebraic structure with a multivalued addition and a standard multiplication.

However, this is not the only approach to the definition of a hyperring. Another important class of hyperrings are *multiplicative hyperrings*. They were introduced in 1982 by R. Rota in [42]. A multiplicative hyperring is a hypercompositional algebraic structure in which the addition is an operation while the multiplication is a hyperoperation. There exist also other variants of hyperrings, e.g. a *feeble hyperring*, where both, addition and multiplication are hyperoperations. They were introduced by C. Corsini in [8]. We refer the reader to [10] for a general survey on hyperring theory.

In this thesis we focus on the hyperrings in the sense of Krasner.

Definition 1.2.1. A *commutative hyperring with unity* is a tuple $(R, +, \cdot, 0, 1)$ which satisfies the following axioms:

(R1) $(R, +, 0)$ is a canonical hypergroup,

(R2) $(R, \cdot, 1)$ is a commutative monoid such that $x \cdot 0 = 0$ for all $x \in R$,

(R3) the operation $\cdot$ is distributive with respect to the hyperoperation $+$, i.e., for all $x, y, z \in R$,

$$x(y + z) = xy + xz,$$

where for $x \in H$ and $A \subseteq R$ we set

$$xA := \{xa \mid a \in A\}.$$

If there is no risk of confusion, we will call $(R, +, \cdot, 0, 1)$ simply a hyperring. If $(R \setminus \{0\}, \cdot, 1)$ is an abelian group, then $(R, +, \cdot, 0, 1)$ is called a *hyperfield*.

We will denote the set of nonzero elements of a hyperring R by R^*.

Remark 1.2.2. The double distributivity law, i.e.,

$$(x + y)(z + t) = xz + xt + yz + yt$$

does not hold in general in hyperrings as we will see in Example 1.2.10. However, the following inclusion is valid in all hyperrings:

$$(x + y)(z + t) \subseteq xz + xt + yz + yt.$$

Indeed, $(x + y)a = xa + ya$ for all $a \in z + t$, so

$$(x + y)(z + t) = \{(x + y)a \mid a \in z + t\} = \{xa + ya \mid a \in z + t\}.$$

Moreover, for every $a \in z + t$,

$$xa + ya \subseteq x(z + t) + y(z + t) = (xz + xt) + (yz + yt) = xz + xt + yz + yt.$$

Hence,

$$(x + y)(z + t) = \{xa + ya \mid a \in z + t\} \subseteq xz + xt + yz + yt.$$

That property was stated without proof in [39] and was shown by O. Viro in [46].

Lemma 1.2.3. *Let R be a hyperring and $A \subseteq R, x, y \in R$. Then*

$$(x + y)A \subseteq xA + yA.$$

Proof. We compute:

$$(x + y)A = \{(x + y)a \mid a \in A\} = \{xa + ya \mid a \in A\} \subseteq \{xa + yb \mid a, b \in A\} = xA + yA.$$

□

Lemma 1.2.4. *Let F be a hyperfield, which is not a field. Then $1 - 1 \neq \{0\}$.*

Proof. Suppose that $1 - 1 = \{0\}$. Then the assumption implies that $x - x = \{0\}$ for every $x \in F$. Take $x, y \in F^*$ and take $a, b \in x + y$. Then $a - b \subseteq (x + y) - (x + y) = (x - x) + (y - y) = 0 + 0 = \{0\}$. Hence, we conclude that $a = b$, so the hyperadditon is single-valued. This shows that F is a field. □

In [23] in 1983 M. Krasner provided the following way of constructing hyperrings and hyperfields. Let R be a commutative ring with unity and let T be a subgroup of its multiplicative semigroup. We consider the following equivalence relation on R:

$$x \sim_T y \Leftrightarrow x = yt \text{ for some } t \in T.$$

We are going to denote the equivalence class of an element $x \in R$ by $[x]_T$ and the set of all equivalence classes by R_T. Observe that $[x]_T = \{xt \mid t \in T\}$. We define the following operations on R_T:

$$[x]_T + [y]_T := \{[xs + yt]_T \mid s, t \in T\},$$

$$[x]_T \cdot [y]_t := [x \cdot y]_T.$$

Theorem 1.2.5 ([23])**.** *The tuple $(R_T, +, \cdot, [0]_T, [1]_T)$ is a hyperring. Moreover, if R is a field, then $(R_T, +, \cdot, [0]_T, [1]_T)$ is a hyperfield.*

Proof. First we will verify axioms (C1)−(C4). Take $[x]_T, [y]_T, [z]_T \in R_T$. Then

$$\begin{aligned}([x]_T + [y]_T) + [z]_T &= \{[xs + yt]_T \mid s, t \in T\} + [z]_T = \{[(xs + yt) + zu]_T \mid s, t, u \in T\} \\ &= \{[xs + (yt + zu)]_T \mid s, t, u \in T\} = [x]_T + \{[yt + zu]_T \mid t, u \in T\} \\ &= [x]_T + ([y]_T + [z]_T),\end{aligned}$$

where we used associativity in R. Hence, axiom (C1) is fulfilled. Axiom (C2) follows from commutativity of addition in R, i.e.,

$$[x]_T + [y]_T = \{[xs + yt]_T \mid s, t \in T\} = \{[yt + xs]_T \mid s, t \in T\} = [y]_T + [x]_T$$

for all $[x]_T, [y]_T \in R_T$. Take now $[x]_T, [y]_T \in R_T$ such that $[0]_T \in [x]_T + [y]_T$. This means that there are $s, t \in T$ such that $0 = xs + yt$. Hence $y = -x(st^{-1})$, which implies that $[y]_T = [-x]_T$. So $-[x]_T := [-x]_T$ is the unique additive inverse of element $[x]_T$. This proves that axiom (C3) holds. Finally, take $[z]_T \in [x]_T + [y]_T$. Then $z = xs + yt$ for some $s, t \in T$ and $y \in -x(st^{-1}) + zt^{-1}$. This shows that $[y]_T \in -[x]_T + [z]_T$, so axiom (C4) is fulfilled. We have proved that R_T is a canonical hypergroup.

Now we need to verify axiom (R2). The fact that multiplication in R_T is associative and commutative follows from the same properties of multiplication in R. Observe that for every $[x]_T \in R_T$, we have $[1]_T \cdot [x]_T = [1 \cdot x]_T = [x]_T$, so $[1]_T$ is a neutral element of multiplication. Moreover, $[x]_T \cdot [0]_T = [x \cdot 0]_T = [0]_T$.

To prove axiom (R3), take $[x]_T, [y]_T, [z]_T \in R_T$. Then

$$\begin{aligned}[x]_T([y]_T + [z]_T) = [x]_T \{[ys + zt]_T \mid s, t \in T\} = \{[x(ys + zt)]_T \mid s, t \in T\} = \\ = \{[xys + xzt]_T \mid s, t \in T\} = [xy]_T + [xz]_T = [x]_T[y]_T + [x]_T[z]_T,\end{aligned}$$

which completes the proof of the fact that $(R_T, +, \cdot, [0]_T, [1]_T)$ is a hyperring.

Assume now that R is a field and take $[0]_T \neq [x]_T \in R_T$. Then $[x^{-1}]_T \in R_T$ is the multiplicative inverse of $[x]_T$. Indeed,

$$[x]_T \cdot [x^{-1}]_T = [xx^{-1}]_T = [1]_T.$$

Hence, in that case, $(R_T, +, \cdot, [0]_T, [1]_T)$ is a hyperfield. □

Remark 1.2.6. A hyperring (hyperfield) which is of the form R_T for some commutative ring with unity (or a field) R is called a quotient hyperring (a quotient hyperfield). However, to distinguish it from the hyperring which arises from the construction described in Proposition 1.3.20, we will call it a *factor hyperring* (or a *factor hyperfield* respectively).

Example 1.2.7. Every field can be seen as a hyperfield if we identify $x + y$ with a singleton $\{x + y\}$. Hence hyperfields can be understood as a generalization of fields.

Example 1.2.8. Consider the set $S := \{-1, 0, 1\}$ with the usual multiplication and the hyperaddition $+$ defined as follows:

$$\begin{aligned}(-1) + (-1) = (-1) + 0 = 0 + (-1) = \{-1\} \\ 0 + 0 = \{0\} \\ 1 + 1 = 1 + 0 = 0 + 1 = \{1\} \\ 1 + (-1) = (-1) + 1 = \{-1, 0, 1\}.\end{aligned}$$

Then $\mathbb{S} := (S, +, \cdot, 0, 1)$ is a hyperfield called the *sign hyperfield*. It can be seen as a factor hyperfield $\mathbb{R}_{(\mathbb{R}^*)^2}$, where $(\mathbb{R}^*)^2$ is the set of nonzero squares of real numbers.

Example 1.2.9. Consider the set $K := \{0, 1\}$ with the usual multiplication and the hyperaddition $+$ defined as follows:

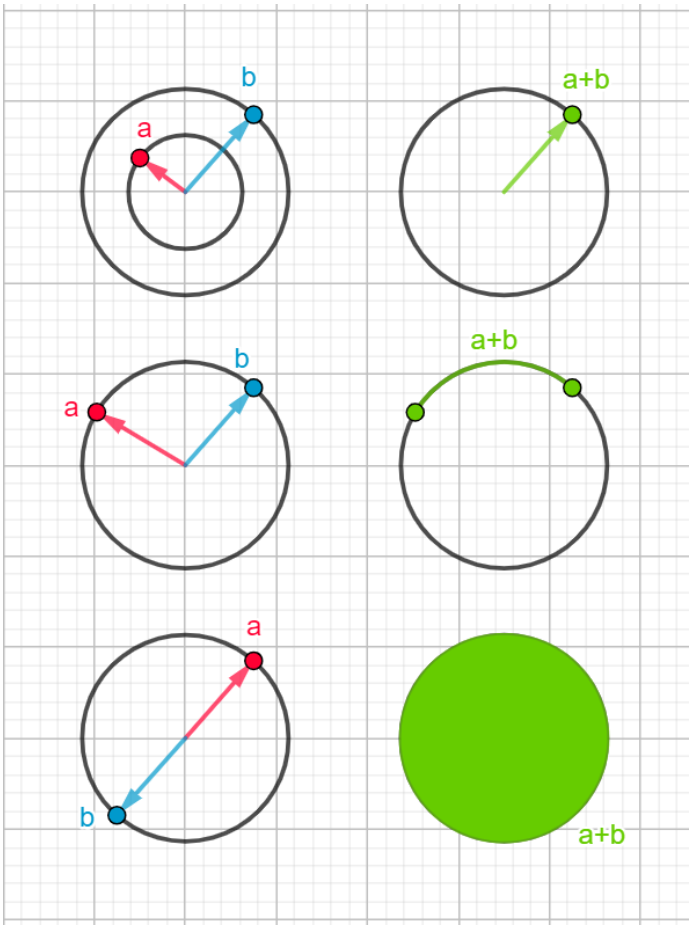
$$\begin{aligned} 0 + 0 &= \{0\} \\ 1 + 0 = 0 + 1 &= \{1\} \\ 1 + 1 &= \{0, 1\}. \end{aligned}$$

Then $\mathbb{K} := (K, +, \cdot, 0, 1)$ is a hyperfield called the *Krasner hyperfield*. It can be seen as a factor hyperfield $\mathbb{C}_{\mathbb{C}^*}$, where $\mathbb{C}^*$ is the set of nonzero complex numbers.

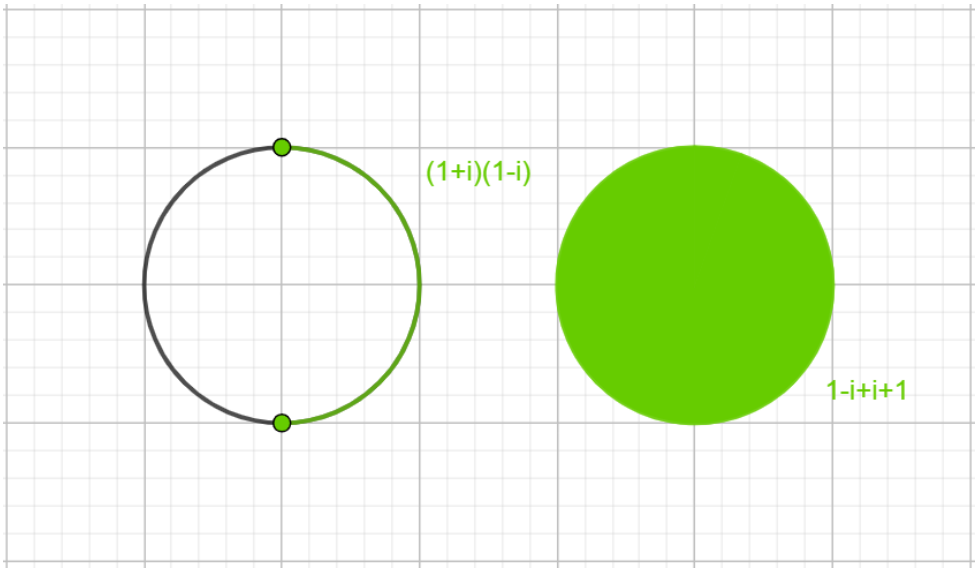
Example 1.2.10 (Definition 6.4, Theorem 6C in [46])**.** We equip the set of complex numbers $\mathbb{C}$ with the usual multiplication and the following hyperaddition:

$$a + b = \begin{cases} a, & \text{if } |a| > |b|, \\ b, & \text{if } |b| > |a|, \\ \text{all the complex numbers} & \\ \text{which belong to the shorter arc} & \text{if } |a| = |b| \text{ and } 0 \neq a +_{\mathbb{C}} b, \\ \text{with endpoints } a \text{ and } b, & \\ \{z \in \mathbb{C} \mid |z| \leq |a|\}, & \text{if } |a| = |b| \text{ and } 0 = a +_{\mathbb{C}} b, \end{cases}$$

where $+_{\mathbb{C}}$ denotes the usual addition of complex numbers.



Then $T\mathbb{C} := (\mathbb{C}, +, \cdot, 0, 1)$ is a hyperfield called the *complex tropical hyperfield*. Note that this hyperfield is not doubly distributive. Indeed, $(1+i)(1-i) \subsetneq 1-i+i+1$. We compute that the result of $1+i$ is the shorter arc with endpoints 1 and i and the result of $1-i$ is the shorter arc with endpoints 1 and $-i$. Hence $(1+i)(1-i)$ is given by the arc between i and $-i$ on the right-hand side of the unit circle, while the result of $1-i+i+1$ is the whole unit circle.



At the end of his paper [23], M. Krasner asked the question whether every hyperring and hyperfield is a factor. The question was answered negatively by Ch. Massouros in [32]. In that paper the author provided the following construction of a hyperfield. In Proposition 1.2.12 we will see an example of a hyperfield constructed in that way which is not a factor hyperfield.

Example 1.2.11 ([32])**.** Let G be any multiplicative group. Let 0 be an additional element and put $0 \cdot x = x \cdot 0 = 0$ for every $x \in G$. Then take $H := G \cup \{0\}$. We define the hyperaddition as follows: for all $x, y \in H$,

$$\begin{aligned} x + 0 = 0 + x &= \{x\} \\ x + y &= \{x, y\} \text{ for } x \neq y \text{ and } x, y \neq 0 \\ x + x &= H \setminus \{x\} \text{ for } x \neq 0. \end{aligned}$$

The tuple $(H, +, \cdot, 0, 1)$ is a hyperfield. The operation of hyperaddition is commutative, so axiom (C2) is fulfilled. To prove the associativity of the hyperaddition we will consider several cases.

- If $x = 0$, then

$$(x + y) + z = (0 + y) + z = y + z = 0 + (y + z) = x + (y + z).$$

  Similarly we can show that the axiom holds if $y = 0$ or $z = 0$.

- If x, y, z are pairwise distinct and nonzero, we have that

$$(x+y)+z = \{x, y\}+z = (x+z)\cup(y+z) = \{x, y, z\} = (x+y)\cup(x+z) = x+\{y, z\} = x+(y+z).$$

- If $x = y \neq z$ and all of the elements are nonzero, then

$$(x + y) + z = (x + x) + z = (H \setminus \{x\}) + z \supseteq (z + z) \cup (0 + z) = (H \setminus \{z\}) \cup \{z\} = H,$$

  Hence, $(x + y) + z = H$. On the other hand,

$$x + (y + z) = x + (x + z) = x + \{x, z\} = (x + x) \cup (x + z) = H.$$

  Similarly we can show the case when $y = z$.

- If $x = z$ and $x \neq y$ and all of the elements are nonzero, then

$$(x + y) + z = (x + y) + x = \{x, y\} + x = H = x + \{x, y\} = x + (y + x) = x + (y + z).$$

- If $x = y = z$ and all of the elements are nonzero, then by commutativity,

$$(x + y) + z = (x + x) + x = x + (x + x) = x + (y + z).$$

The unique inverse of $x \in H$ is $x \in H$, hence $x = -x$. To see that axiom (C4) holds, we need to consider several cases. Take $x, y \in H$.

- If $x = y = 0$, then $z \in x + y = 0 + 0 = \{0\}$ implies that $z = 0$. Hence $y \in -x + z$.
- If $x \neq y$ and $x, y \neq 0$, then $z \in x + y$ implies that $z = x$ or $z = y$. If $z = x$, then $y \in H \setminus \{x\} = x + x = -x + z$. If $z = y$, then $y \in \{x, y\} = x + y = -x + y = -x + z$.
- If $x = y \neq 0$, then $z \in x + y = x + x = H \setminus \{x\}$ implies that $z \neq x$. Hence $x \in \{x, z\} = x + z = -x + z$.
- If $x = 0$ and $y \neq 0$, then $z \in x + y = 0 + y = \{y\}$ implies $z = y$. Hence $y \in \{y\} = 0 + y = 0 + z = -x + z$.
- If $y = 0$ and $x \neq 0$, then $z \in x + y = x + 0 = \{x\}$ implies $z = x$. So $0 \in H \setminus \{x\} = x + x = -x + x = -x + z$.

The axiom (R2) follows from the fact that G is a multiplicative group. It is left to show distributivity. Take $x, y, z \in H$.

- If $y \neq z$, then

$$x(y + z) = x \cdot \{y, z\} = \{xy, xz\} = xy + xz,$$

  since $y \neq z$ implies $xy \neq xz$.

- If $y = z$, then

$$x(y + z) = x(y + y) = x \cdot (H \setminus \{y\}) = H \setminus \{xy\} = xy + xy = xy + xz.$$

This finishes the proof.

We will call this hyperfield a *Massouros hyperfield*.

The following proposition can be found in [32]. We present the proof of that result, filling a gap that appeared in the original paper.

Proposition 1.2.12 ([32]). *There exists a Massouros hyperfield which is not a factor hyperfield.*

Proof. Consider a multiplicative group G such that $|G| > 2$ and $x^2 = 1$ for every $x \in G$. Let $H := G \cup \{0\}$ be a hyperfield as described in the Example 1.2.11. From our assumption, $|H| > 3$. Suppose that H is a factor hyperfield K_T. Then the following properties are valid:

(1) $[x]_T \cdot [x]_T = [x^2]_T = [1]_T$ for $[x]_T \neq [0]_T$, i.e., $x \neq 0$ implies that $x^2 \in T$,

(2) $[x]_T + [x]_T = K_T \setminus \{[x]_T\}$ for $[x]_T \neq [0]_T$,

(3) $[1]_T = -[1]_T$, in other words $T = -T$,

(4) $T + T := \{t_1 + t_2 \mid t_1, t_2 \in T\} = K \setminus T$.

The last property needs more explanation. Take any $x \in T + T$. Then there are $t_1, t_2 \in T$ such that $x = t_1 + t_2$. This means that $[x]_T \in [1]_T + [1]_T = K_T \setminus \{[1]_T\}$. Hence $[x]_T \neq [1]_T$, which is equivalent to $x \in K \setminus T$. Moreover, note that from (1) it follows that every nonzero square is in T.

We are going to show that H is not a factor hyperfield. Assume first that $\operatorname{char} K \neq 2$. Then any nonzero $a \in K$ can be written in the following form:

$$a = (\frac{a+1}{2})^2 - (\frac{a-1}{2})^2.$$

These squares are nonzero if $a \neq 1, -1$. Hence, from what we have observed before, for $a \neq 1, -1$ we obtain that $(\frac{a+1}{2})^2, (\frac{a-1}{2})^2 \in T$. So $a \in T - T = T + T = K \setminus T$, where we used properties (3) and (4). We conclude that $T = \{-1, 1\}$.

From the definition of the hyperaddition in H, for $x \notin T$ and $x \neq 0$ we have

$$[1]_T + [x]_T = \{[1]_T, [x]_T\}.$$

On the other hand, from the definition of the hyperaddition in a factor hyperfield and $T = \{-1, 1\}$, we obtain

$$[1]_T + [x]_T = \{[1 - x]_T, [1 + x]_T\}.$$

Hence, we have two cases:

$$[1 + x]_T = [x]_T \text{ and } [1 - x]_T = [1]_T \qquad \text{or} \qquad [1 + x]_T = [1]_T \text{ and } [1 - x]_T = [x]_T.$$

In the first case we obtain $1+x=-x$ and $1-x=-1$, so $2x=-1$ and also $x=2$. In the second case, $1+x=-1$ and $1-x=x$, so $x=-2$ and at the same time $2x=1$. In both cases we obtain that $\operatorname{char} K=5$. Observe that since the polynomial $(X^2-1)(X^2+1)$ admits at most four roots and every nonzero element of H is a root of that polynomial, we conclude that $K=\mathbb{F}_5$. Hence $H=\{[0]_T,[1]_T,[2]_T\}$, which is a contradiction to the assumption that $|H|>3$.

Assume now that $\operatorname{char} K=2$. If $T=\{-1,1\}$ then from (1) for every $x\neq 0$ we have $x^2=1=-1$. Hence $x=1$ and $K=\mathbb{F}_2$, so $H=\{[0]_T,[1]_T\}$. This is a contradiction with the assumption that $|H|>3$. Thus, there exists $a\in T$ such that $a\neq -1,1$. Consider the element $b:=a+1$ and compute

$$1=a^2+a^2+1=a^2+(a+1)^2=a^2+b^2\in T+T=K\setminus T.$$

Since $1\in T$, we obtain a contradiction. Hence we conclude that H is not a factor hyperfield. □

Another important and interesting class of hyperfields are *stringent hyperfields*. They were defined by N. Bowler and T. Su in [7] in the following way:

Definition 1.2.13 ([7])**.** A canonical hypergroup G is stringent if for $x,y\in G$ the set $x+y$ is a singleton unless $0\in x+y$. A hyperfield F is stringent if its underlying hypergroup is stringent.

Example 1.2.14. The sign hyperfield $\mathbb{S}$ and the Krasner hyperfield $\mathbb{K}$ are stringent.

As we have already mentioned in Remark 1.2.2, not every hyperfield satisfies the double distributive law. However, we have the following result for double distributive hyperfields.

Proposition 1.2.15 (Proposition 1.3, [7])**.** *Every hyperfield, in which the double distributive law holds, is stringent.*

Proof. Let F be a double distributive hyperfield and take $a,b\in F^*$ such that $0\notin a+b$. Take $x,y\in a+b$. Then we compute

$$0\in 1-1\subseteq x\cdot x^{-1}-y\cdot y^{-1}\subseteq (a+b)\cdot x^{-1}-(a+b)\cdot y^{-1}=(a+b)(x^{-1}-y^{-1}).$$

Since $0\notin a+b$, we have $0\in x^{-1}-y^{-1}$. Hence $x^{-1}=y^{-1}$, so $x=y$. This proves that F is stringent. □

In their paper, N. Bowler and T. Su provided a classification of stringent hyperfields. Their work was continued by J. Maxwell and B. Smith. We keep the notation introduced by them in [36]. Let $(\mathbb{H},+_{\mathbb{H}},\cdot_{\mathbb{H}},0_{\mathbb{H}},1_{\mathbb{H}})$ be a hyperfield and $(G,+_G,0_G)$ a totally ordered abelian group. Define $F=\mathbb{H}\rtimes G:=((\mathbb{H}^*\times G)\cup\{0\},+,\cdot,0,(1_{\mathbb{H}},0_G))$. This tuple is a hyperfield, where:

$$0\cdot x=x\cdot 0=0 \text{ for every } x\in F,$$

$$(x,g)\cdot(y,h)=(x\cdot_{\mathbb{H}} y,\ g+_G h) \text{ for any } (x,g),(y,h)\in\mathbb{H}^*\times G.$$

The hyperaddition is defined as follows:
$x + 0 = 0 + x = \{x\}$ for every $x \in F$,

$$(x,g)+(y,h) = \begin{cases} (x,g), & \text{if } g > h \\ (y,h), & \text{if } g < h \\ \{(z,g) \mid z \in x +_{\mathbb{H}} y\}, & \text{if } g = h \text{ and } 0_{\mathbb{H}} \notin x +_{\mathbb{H}} y \\ \{(z,g) \mid z \in (x +_{\mathbb{H}} y) \cap \mathbb{H}^*\} \cup \{(c,g') \mid c \in \mathbb{H}^*, g' < g\} & \text{if } g = h \text{ and } 0_{\mathbb{H}} \in x +_{\mathbb{H}} y. \end{cases}$$

The next two theorems are the main results of the paper [7] of Bowler and Su. For the proofs we refer the reader to the original paper.

Theorem 1.2.16 (Theorem 4.10, [7]). *Every stringent hyperfield is of the form $\mathbb{H} \rtimes G$, where $\mathbb{H}$ is either the Krasner hyperfield $\mathbb{K}$ or the sign hyperfield $\mathbb{S}$ or a field.*

Theorem 1.2.17 (Theorem 7.5, [7]). *Every stringent hyperfield is a factor hyperfield.*

1.3 Elements of hyperring theory

1.3.1 Subhyperrings and homomorphisms

In the literature there are two different approaches to the definition of a subhyperring of a given hyperring R with a hyperaddition $+$. In [18] Jun defined a subhyperring as a subset S of R which, with *induced* operations, is a hyperring itself. We can also require, like in the case of rings, that $a + (-b) \subseteq S$ for every $a, b \in S$ (see e.g. [11], [40]). Here we distinguish the two approaches giving the following definition.

Definition 1.3.1. Let $(R, +, \cdot, 0)$ be a hyperring.

(a) A nonempty subset $S \subseteq R$ is called a *subhyperring* if $0 \in S$, for every $a, b \in S$ one has $ab \in S$, and with the induced operation

$$a +_S b := (a + b) \cap S,$$

the tuple $(S, +_S, \cdot, 0)$ is a hyperring.

(b) The tuple $(S, +, \cdot, 0)$ is called a *strict subhyperring* if $a - b \subseteq S$ and $ab \in S$ for every $a, b \in S$.

Remark 1.3.2. A strict subhyperring is indeed a hyperring. Since S is nonempty, there exists $a \in S$. Then $0 \in a - a \subseteq S$. Hence, for every $a \in S$ we have that $\{-a\} = 0 - a \subseteq S$. This shows that axiom (C3) from Definition 1.1.7 is fulfilled. All of the other axioms follow directly from the same axioms in R. Moreover, we observe that every strict subhyperring is a subhyperring, but the converse is not true as we will see in Example 2.2.9.

Lemma 1.3.3. *Let F be a hyperring. The intersection of strict subhyperrings of F is a strict subhyperring of F.*

Proof. Let R be a hyperring and consider $\bigcap_{j\in J} S_j$, where each S_j is a strict subhyperring in R and J is any index set. Take $a, b \in \bigcap_{j\in J} S_j$. Then $a, b \in S_j$ for every $j \in J$, so $a - b \subseteq S_j$ and $ab \in S_j$ for every $j \in J$. Hence $a - b \subseteq \bigcap_{j\in J} S_j$ and $ab \in \bigcap_{j\in J} S_j$. We conclude that $\bigcap_{j\in J} S_j$ is a strict subhyperring of F. □

Definition 1.3.4. Let R and S be hyperrings and $\varphi : R \to S$ be a map such that:

(HH1) $\varphi(0_R) = 0_S$,

(HH2) $\varphi(x \cdot_R y) = \varphi(x) \cdot_S \varphi(y)$,

(HH3) $\varphi(x +_R y) \subseteq \varphi(x) +_S \varphi(y)$.

Then φ is called a *homomorphism of hyperrings*. If φ satisfies condition

(HH3') $\varphi(x +_R y) = \varphi(x) +_S \varphi(y)$,

which is stronger than (HH3), then φ is called a *strict homomorphism of hyperrings*.

Example 1.3.5. Consider a field K with a multiplicative subgroup $T \subseteq K^*$ and the factor hyperfield K_T. Then the map $\eta : K \to K_T$ defined as $x \mapsto [x]_T$ for $x \in K^*$ and $0 \mapsto [0]_T$ is a hyperfield homomorphism. Indeed, axiom (HH1) follows from the definition of η. Axiom (HH2) is a consequence of the definition of multiplication in K_T:

$$\eta(x \cdot y) = [xy]_T = [x]_T \cdot [y]_T = \eta(x) \cdot \eta(y).$$

Finally, observe that

$$\eta(x + y) = [x + y]_T \in [x]_T + [y]_T = \eta(x) + \eta(y),$$

so axiom (HH3) is also fulfilled.

Proposition 1.3.6. *Let $\varphi : R \to S$ be a homomorphism of hyperrings, which is not the zero homomorphism. Then*

(a) $\varphi(1_R) = 1_S$,

(b) $\varphi(-x) = -\varphi(x)$ *for every* $x \in R$.

Proof. (a) From axiom (HH2) we have

$$\varphi(1_R) = \varphi(1_R \cdot 1_R) = \varphi(1_R) \cdot \varphi(1_R).$$

Since φ is not the zero homomorphism, we know that $\varphi(1_R) \neq 0_S$, so $\varphi(1_R) = 1_S$.

(b) From axioms (HH1) and (HH3) we obtain

$$0_S = \varphi(0_R) \in \varphi(x - x) \subseteq \varphi(x) + \varphi(-x).$$

From axiom (C3) of Definition 1.1.7 we have that the inverse is unique, hence $\varphi(-x) = -\varphi(x)$. □

Definition 1.3.7. Let $\varphi : R \to S$ be a bijective, strict homomorphism of hyperrings. Then φ is called an *isomorphism of hyperrings* and R and S are called *isomorphic*, which is denoted by $R \simeq S$.

Definition 1.3.8. Let $\varphi : R \to S$ be a homomorphism of hyperrings.

(a) The set $\ker \varphi := \{x \in R \mid \varphi(x) = 0_S\}$ is called the *kernel* of φ.

(b) The set $\mathrm{Im}\varphi := \{\varphi(x) \mid x \in R\}$ is called the *image* of φ.

1.3.2 Hyperideals

In this thesis we consider only commutative hyperrings. Hence we do not distinguish right and left hyperideals. However, these structures have been considered e.g. in Chapter 3.2 of [10].

Definition 1.3.9. Let R be a hyperring.

(a) A nonempty subset $I \subseteq R$ is a *hyperideal* if it is a strict subhyperring such that $ra \in I$ for all $r \in R$ and $a \in I$. We say that a hyperideal I is *proper* if $I \neq R$.

(b) A proper hyperideal $I \subsetneq R$ is *prime* if $ab \in I$ implies $a \in I$ or $b \in I$.

(c) A proper hyperideal $I \subsetneq R$ is *maximal* if I satisfies the following property: if $J \subseteq R$ is a hyperideal of R such that $I \subsetneq J$, then $J = R$.

Lemma 1.3.10. *The intersection of hyperideals in a hyperring R is again a hyperideal in R.*

Proof. Using Lemma 1.3.3 we know that the intersection of hyperideals in a hyperring R is a strict subhyperring in R. Moreover, take $a \in \bigcap_{j \in J} I_j$, where each I_j is a hyperideal in R and J is any index set. Take $r \in R$ and consider the element ra. Since $a \in I_j$ for every $j \in J$, we obtain $ra \in I_j$ for every $j \in J$. This implies that $ra \in \bigcap_{j \in J} I_j$, which completes the proof. □

Lemma 1.3.11. *The union of a chain of hyperideals in a hyperring R is again a hyperideal in R.*

Proof. Let R be a hyperring and let J be an ordered set. Let C be a chain $(I_i)_{i \in J}$ of hyperideals in R such that $I_{i_1} \subseteq I_{i_2}$ if $i_1 < i_2$. Take $a, b \in \bigcup_{i \in J} I_i$. Then there are $i_1, i_2 \in J$ such that $a \in I_{i_1}$ and $b \in I_{i_2}$. Let $j := \max\{i_1, i_2\}$. Then $a, b \in I_j$ and since I_j is a strict subhyperring, we obtian that $a - b \subseteq I_j$. Hence $a - b \subseteq \bigcup_{i \in J} I_i$. Take now $a \in \bigcup_{i \in J} I_i$ and $r \in R$. Then there is $j \in J$ such that $a \in I_j$. We have $ra \in I_j$, so $ra \in \bigcup_{i \in J} I_i$. This proves that $\bigcup_{i \in J} I_i$ is a hyperideal in R. □

Lemma 1.3.12. *Let R, S be hyperrings and $\varphi : R \to S$ be a homomorphism of hyperrings. Let I be a hyperideal in S. Then $\varphi^{-1}(I)$ is a hyperideal in R.*

Proof. Take $a, b \in \varphi^{-1}(I)$. By axiom (HH3) and part (b) of Proposition 1.3.6 we have

$$\varphi(a -_R b) \subseteq \varphi(a) -_S \varphi(b) \subseteq I.$$

Hence $a -_R b \subseteq \varphi^{-1}(I)$. Now take $r \in R$. Then

$$\varphi(r \cdot_R a) = \varphi(r) \cdot_S \varphi(a) \in I,$$

where we used axiom (HH2). So $r \cdot_R a \in \varphi^{-1}(I)$. Moreover, $\varphi(0_R) = 0_S \in I$ by axiom (HH1), so $0_R \in \varphi^{-1}(I)$. Hence $\varphi^{-1}(I)$ is a hyperideal in R. □

Definition 1.3.13. (a) Let R be a hyperring and $a, b \in R^*$. If $a \cdot b = 0$, then a and b are called *zero divisors* in R. A hyperring R with no zero divisors is called an *integral hyperdomain*.

(b) Let R be a hyperring. We say that $a \in R$ is a *unit* if there exists $b \in R$ such that $a \cdot b = 1$. We will denote by $R^\times$ the set of all units in R.

The statements of the following lemmas can be found in the literature. However, in the papers there are no proofs of these results, so we prove them here:

Lemma 1.3.14 (Lemma 2.12, [26])**.** *Let F be a hyperring and I a hyperideal. If I contains a unit, then $I = R$.*

Proof. Let $u \in R^\times \cap I$. Then for every $r \in R$ we have that $r = u \cdot (u^{-1}r) \in I$, so $R \subseteq I$. □

Corollary 1.3.15 (Corollary 2.13, [26])**.** *The only hyperideals of a hyperfield F are $\{0\}$ and F.*

Lemma 1.3.16 (Proposition 2.12, [11])**.** *Let R be a hyperring and $I \subseteq R$ a proper hyperideal. Then I is contained in some maximal hyperideal $\mathcal{M}$ of R.*

Proof. Consider the following set of hyperideals

$$S(I) := \{I \subseteq J \mid J \text{ is a proper hyperideal of } R\}.$$

Take any chain C of ideals from $S(I)$. Then the union of hyperideals $\bigcup_{J \in C} J$ is again an element from $S(I)$. Indeed, a union of a chain of hyperideals is again a hyperideal by Lemma 1.3.11. Moreover, $I \subseteq \bigcup_{J \in C} J$ and $1 \notin \bigcup_{J \in C} J$, because $1 \notin J$ for every $J \in C$. Hence every chain from $S(I)$ has an upper bound in $S(I)$, so from Zorn's Lemma we obtain that there exists at least one maximal ideal $\mathcal{M} \in S(I)$. □

Lemma 1.3.17 (Lemma 4.3.4, [19], Lemma 3.22, [18])**.** *Let J be an index set and R a hyperring.*

(a) *Let $a \in R$. Then the set*

$$\langle a \rangle := \{ra \mid r \in R\}$$

is a hyperideal in R.

(b) *Assume that $I_j = \langle a_j \rangle$ is a hyperideal for each $j \in J$. Then*

$$\langle I_j \rangle_{j \in J} := \{x \in \sum_{j \in X} r_j a_j \mid r_j \in R, X \subseteq J, |X| < \infty\}$$

is a hyperideal.

(c) *Let $\{I_j\}_{j \in J}$ be a family of hyperideals of R. Then*

$$\langle I_j \rangle_{j \in J} := \{x \in \sum_{j \in X} r_j a_j \mid r_j \in R, a_j \in I_j, X \subseteq J, |X| < \infty\}$$

is a hyperideal.

Proof. (a) Take $a \in R$ and $ra, sa \in \langle a \rangle$, where $r, s \in R$. Then $ra - sa = (r - s)a \subseteq \langle a \rangle$, because $r - s \subseteq R$. Moreover, $ra \cdot sa = (ras)a \in \langle a \rangle$, since $ras \in R$. Hence $\langle a \rangle$ is a strict subhyperring in R. Take any $s \in R$ and $ra \in \langle a \rangle$. Then $s(ra) = (sr)a \in \langle a \rangle$, where we used associativity of multiplication in R. So $\langle a \rangle$ is a hyperideal in R.

(c) Take $x, y \in \langle I_j \rangle_{j \in J}$. Then there are $a_{i_1}, \ldots, a_{i_n}, b_{k_1}, \ldots, b_{k_m}$ with $a_i \in I_i, b_k \in I_k$ for $i \in \{i_1, \ldots, i_n\}$, $k \in \{k_1, \ldots, k_m\}$ and $r_{i_1}, \ldots, r_{i_n}, s_{k_1}, \ldots, s_{k_m} \in R$ such that $x \in r_{i_1}a_{i_1} + ...r_{i_n}a_{i_n}$ and $y \in s_{k_1}b_{k_1} + ... + s_{k_m}b_{k_m}$. Let $T := \{i_1, \ldots, i_n, k_1, \ldots, k_m\} \subseteq J$ with $|T| =: p \leq n + m$. We can write $x \in r_{t_1}a_{t_1} + ... + r_{t_p}a_{t_p}$ and $y \in s_{t_1}b_{t_1} + ... + s_{t_p}b_{t_p}$, where $t_i \in T$. Then

$$x - y \subseteq r_{t_1}a_{t_1} + ... + r_{t_p}a_{t_p} - (s_{t_1}b_{t_1} + ... + s_{t_p}b_{t_p}) = (r_{t_1}a_{t_1} - s_{t_1}b_{t_1}) + ... + (r_{t_p}a_{t_p} - s_{t_p}b_{t_p}).$$

Consider $z \in x - y$. Then there are $c_{t_i} \in r_{t_i}a_{t_i} - s_{t_i}b_{t_i} \subseteq I_{t_i}$ such that $z \in c_{t_1} + ... + c_{t_p}$, which proves that $z \in \langle I_j \rangle_{j \in J}$. Hence $x - y \subseteq \langle I_j \rangle_{j \in J}$. Moreover, for any $u \in R$ we have

$$ux \in u(r_{i_1}a_{i_1} + ...r_{i_n}a_{i_n}) = (ur_{i_1})a_{i_1} + ... + (ur_{i_n})a_{i_n},$$

so $ux \in \langle I_j \rangle_{j \in J}$. This proves that $\langle I_j \rangle_{j \in J}$ is a hyperideal in R.

(b) It is a special case of part (c).

□

Remark 1.3.18. If $a \in R \setminus R^\times$, then the hyperideal $\langle a \rangle$ is a proper hyperideal.

1.3.3 Quotient hyperrings

We wish to give a contruction of a quotient hyperring, analogous to the classical construction of quotient ring. The construction has been presented by B. Davvaz and A. Salasi in [11] for quotients of, so called, *normal hyperideals*. Later, it was generalized by J. Jun in [18] to every hyperideal.

Let R be a hyperring and I a hyperideal. Let us consider the following equivalence relation on R:

$$x \sim_I y :\Leftrightarrow x + I = y + I, \tag{1.1}$$

where $x + I := \bigcup_{a \in I}(x + a)$.

Lemma 1.3.19 (Lemma 3.3, [18]). *Let R be a hyperring and I a hyperideal. Then for all $x, y \in R$,*

$$x \sim y \Leftrightarrow (x - y) \cap I \neq \emptyset.$$

Proof. Assume first that $x \sim_I y$, i.e., $x + I = y + I$. Since $0 \in I$, we have $\{x\} = x + 0 \subseteq y + I$. This means that there exists $a \in I$ such that $x \in y + a$. From the reversibility axiom, $a \in x - y$, so $a \in (x - y) \cap I$. Hence, $(x - y) \cap I \neq \emptyset$.

Assume now that $(x - y) \cap I \neq \emptyset$. This means that there is $a \in (x - y) \cap I$. From the reversibility axiom, $x \in a + y$. Take $z \in x + I$. Then there exists $b \in I$ such that $z \in x + b$. We compute

$$z \in x + b \subseteq (a + y) + b = y + (a + b) \subseteq y + I,$$

where we used the fact that I is a strict subhyperring of R. Hence $x + I \subseteq y + I$. Similarly we show that $y + I \subseteq x + I$, so $x \sim_I y$. □

Proposition 1.3.20 (Proposition 3.5, [18]). *Let R be a hyperring and I a hyperideal, and let R/I be the set of equivalence classes of the relation (1.1). We define operations on R/I as follows:*

$$(a + I) + (b + I) := \{c + I \mid c \in a + b\},$$
$$(a + I) \cdot (b + I) := ab + I.$$

Then the tuple $(R/I, +, \cdot, 0 + I)$ is a hyperring.

Proof. First we will show that operations on R/I are well-defined. Assume that $a_1 + I = a_2 + I$ and $b_1 + I = b_2 + I$. We are going to show that $(a_1 + I) + (b_1 + I) = (a_2 + I) + (b_2 + I)$. Take $c + I \in (a_1 + I) + (b_1 + I)$. Then $c \in a_1 + b_1$. From reversibility axiom, we have $a_1 \in c - b_1$. Since $a_1 + I = a_2 + I$, there exists $x \in (a_1 - a_2) \cap I$. We compute

$$x \in a_1 - a_2 \subseteq c - b_1 - a_2 = c - (b_1 + a_2).$$

Then there exists $y \in b_1 + a_2$ such that $x \in (c - y) \cap I$. We conclude that $c + I = y + I$, so $(a_1+I)+(b_1+I) \subseteq (a_2+I)+(b_1+I)$. Similarly, we can show that $(a_2+I)+(b_1+I) \subseteq (a_1+I)+(b_1+I)$.

Next, we can show that $(a_2+I)+(b_1+I)=(a_2+I)+(b_2+I)$. This completes the argument that $(a_1+I)+(b_1+I)=(a_2+I)+(b_2+I)$.

Now assume again that $a_1+I=a_2+I$ and $b_1+I=b_2+I$. We wish to show that $(a_1+I)\cdot(b_1+I)=(a_2+I)\cdot(b_2+I)$. Take $x\in(a_1-a_2)\cap I$. Hence, $xb_1\in(a_1-a_2)b_1=a_1b_1-a_2b_1$, so $(a_1b_1-a_2b_1)\cap I\neq\emptyset$. We conclude that $(a_1+I)(b_1+I)=a_1b_1+I=a_2b_1+I=(a_2+I)(b_1+I)$. Similarly, we show that $(a_2+I)(b_1+I)=(a_2+I)(b_2+I)$. This proves that $(a_1+I)(b_1+I)=(a_2+I)(b_2+I)$. We have shown that the operations are well-defined.

The set R/I inherits commutativity from R. Hence axiom (C2) from Definition 1.1.7 is fulfilled. Moreover, for $a+I$ we observe that $0+I\in(a+I)+(-a+I)$. Suppose that there exists another $b+I\in R/I$ such that $0+I\in(a+I)+(b+I)$. Then $0\in a+b$, so $b=-a$ by axiom (C3) in R. Hence, axiom (C3) holds in R/I. Take now $c+I\in(a+I)+(b+I)$. Then $c\in a+b$. From the reversibility axiom in R we have that $b\in -a+c$. So $b+I\in(-a+I)+(c+I)$, which proves that axiom (C4) holds in R/I. It is left to show that axiom (C1) is fulfilled. We compute

$$\begin{aligned}((a+I)+(b+I))+(c+I)&=\{x+I\mid x\in a+b\}+(c+I)=\{y+I\mid x\in a+b,\ y\in x+c\}\\&=\{y+I\mid y\in(a+b)+c\}=\{y+I\mid y\in a+(b+c)\}\\&=(a+I)+\{z+I\mid z\in b+c\}=(a+I)+((b+I)+(c+I)).\end{aligned}$$

So R/I is a canonical hypergroup.

Multiplication in R/I is commutative. Moreover, $(a+I)(1+I)=a+I$ and $(a+I)(0+I)=0+I$. Hence, axiom (R2) holds. Finally, we observe

$$\begin{aligned}(a+I)\cdot((b+I)+(c+I))&=(a+I)\cdot\{x+I\mid x\in b+c\}=\{ax+I\mid x\in b+c\}\\&=\{y+I\mid y\in ab+ac\}=(ab+I)+(ac+I),\end{aligned}$$

so axiom (R3) is fulfilled. This finishes the proof. □

Definition 1.3.21. The hyperring $(R/I,+,\cdot,0+I,1+I)$ defined above is called a *quotient hyperring*.

Proposition 1.3.22 (Proposition 3.6, [18])**.** *Let R be a hyperring and I a hyperideal. The* projection

$$\begin{aligned}\pi:R&\to R/I\\x&\mapsto x+I\end{aligned}$$

is a strict, surjective homomorphism of hyperrings and $\ker\pi=I$.

Proof. We compute $\pi(0_R)=0+I=0_{R/I}$, so axiom (HH1) holds. Next, $\pi(x\cdot y)=xy+I=(x+I)\cdot(y+I)=\pi(x)\cdot\pi(y)$. Hence, axiom (HH2) is fulfilled. Moreover, observe that

$$\pi(x+y)=\pi(\{c\mid c\in x+y\})=\{c+I\mid c\in x+y\}=(x+y)+(y+I),$$

so axiom (HH3') holds. This proves that π is a strict homomorphism of hyperrings. Take any $x + I \in R/I$. Then $\pi(x) = x + I$, so π is surjective. Finally,

$$\ker \pi = \{x \mid \pi(x) = 0_{R/I}\} = \{x \mid x + I = 0 + I\} = \{x \mid x \in I\} = I.$$

□

Proposition 1.3.23 (Proposition 2.11, [26]). *Let R be a hyperring. The kernel of every homomorphism of hyperrings $\varphi : R \to S$ is a hyperideal of R. Every hyperideal of R is the kernel of some strict homomorphism of hyperrings.*

Proof. Take $x, y \in \ker \varphi$. Then $\varphi(x) = 0_{R/I}$ and $\varphi(y) = 0_{R/I}$. From axiom (HH3) from Definition 1.3.4 we obtain

$$\varphi(x +_R y) \subseteq \varphi(x) +_S \varphi(y) = 0_S + 0_S = 0_S,$$

so $x +_R y \subseteq \ker \varphi$. Moreover, from axiom (H2) we have that for any $r \in R$,

$$\varphi(rx) = \varphi(r) \cdot_S \varphi(x) = \varphi(r) \cdot_S 0_S = 0_S.$$

Hence, $rx \in \ker \varphi$. Finally, $\varphi(0_R) = 0_S$ by axiom (HH1), so $0_R \in \ker \varphi$. This proves that $\ker \varphi$ is a hyperideal in R.

Take any hyperideal I in R and consider the quotient hyperring R/I. Then I is a kernel of the canonical projection $R \to R/I$, which is a strict, surjective hyperring homomorphism by Proposition 1.3.22. □

Proposition 1.3.24 (Proposition 2.16, [26]). *Let R be a hyperring and I a hyperideal.*

(a) I is prime if and only if R/I is an integral hyperdomain.

(b) I is maximal if and only if R/I is a hyperfield.

Proof. (a) Take $x + I, y + I \in R/I$ such that $(x + I)(y + I) = 0 + I$. Then $(xy - 0) \cap I \neq \emptyset$, so $xy \in I$. The hyperideal I is prime if and only if $x \in I$ or $y \in I$, which is equivalent to $x + I = 0 + I$ or $y + I = 0 + I$. The latter is true if and only if R/I is an integral hyperdomain.

(b) Let I be a maximal hyperideal. We wish to show that every nonzero $x + I \in R/I$ has a multiplicative inverse. Since $x + I$ is nonzero, $x \notin I$. Consider a hyperideal J constructed as in Lemma 1.3.17 (c) using hyperideals I and $\langle x \rangle$. Clearly $I \subsetneq J$. Since I is maximal, we conclude $J = R \ni 1$. By definition of J we obtain that

$$1 \in xy + i$$

for some $y \in R$ and $i \in I$. From the reversibility axiom we have $i \in 1 - xy$. Hence $i \in (1 - xy) \cap I \neq \emptyset$. From Lemma 1.3.19 we obtain that $1 + I = xy + I = (x + I)(y + I)$, which means that $y + I$ is a multiplicative inverse of $x + I$.

To prove the converse, assume that R/I is a hyperfield and consider a hyperideal J such that $I \subsetneq J$. Take $x \in J \setminus I$. Then $x + I \neq 0 + I$, so there exists $y + I \in R/I$ such that

$$xy + I = (x + I)(y + I) = 1 + I.$$

Hence $\emptyset \neq (xy - 1) \cap I \subseteq (xy - 1) \cap J$. Since $x \in J$, it is true that $xy \in J$. Therefore, $0 + J = xy + J = 1 + J$, so $1 \in J$. This, by Lemma 1.3.14, implies that $J = R$. □

The last proposition gives us a different proof of the following result, which has been proven in [11]:

Corollary 1.3.25. *Every maximal hyperideal of a hyperring R is a prime hyperideal in F.*

Proof. Let I be a maximal hyperideal in a hyperring R. Then the quotient hyperring R/I is a hyperfield by Proposition 1.3.24 (b). Since every hyperfield is a hyperdomain, we obtain, from Proposition 1.3.24 (a), that I is prime. □

1.3.4 Hyperring of fractions and localization

The following construction can be found in [11]. Let R be a hyperring and $S \subseteq R^*$ a multiplicatively closed subset with 1. The relation $\sim_S$ defined on $R \times S$ by

$$(a, s) \sim_S (b, t) \Leftrightarrow \exists_{u \in S} \ \ uta = usb.$$

is an equivalence relation. Indeed, it is obvious that $\sim_S$ is reflexive and symmetric. We are going to show that $\sim$ is transitive. Take $(a, s), (b, t), (c, y) \in R \times S$ such that $(a, s) \sim_S (b, t)$ and $(b, t) \sim_S (c, y)$. This means that there exist $u, w \in S$ such that

$$uat = ubs \text{ and } wby = wct.$$

Then

$$(tuw)ay = (wy)uat = (wy)ubs = (su)wby = (su)wct = (tuw)cs.$$

Since $t, u, w \in S$ and S is multiplicatively closed, we obtain that $(a, s) \sim_S (c, y)$.

In $R \times S$ the equivalence class containing (a, s) is denoted by $\frac{a}{s}$ and the set of all equivalence classes by $\frac{R}{S}$. Having this notation we can write

$$\frac{a}{s} = \frac{b}{t} \Leftrightarrow \exists_{u \in S} \ \ uta = usb.$$

Lemma 1.3.26. *Let R be a hyperring and $S \subseteq R^*$ a multiplicatively closed subset with 1. Consider the set $\frac{R}{S}$. Then $\frac{ax}{sx} = \frac{a}{s}$ for every $a \in R$ and $s, x \in S$.*

Proof. By associativity and commutativity of the multiplication in R we have $1(ax)s = 1a(sx)$. Since $1 \in S$, we obtain that $\frac{ax}{sx} = \frac{a}{s}$. □

The next theorem can be found in [11]. We will show the proof of this result giving all the details.

Proposition 1.3.27 (Theorem 3.1, [11]). *Let $(R,+,\cdot,0,1)$ be a hyperring and $S \subseteq R^*$ a multiplicatively closed subset with 1. Then $\frac{R}{S}$ with operations*

$$\frac{a}{s}+\frac{b}{t} := \{\frac{y}{st} \mid y \in at+bs\},$$

$$\frac{a}{s}\cdot\frac{b}{t} := \frac{ab}{st}$$

is a hyperring.

Proof. First of all we will show that the operations + and $\cdot$ on $\frac{R}{S}$ are well-defined. Take $\frac{a}{s}, \frac{a_1}{s_1}, \frac{b}{t}, \frac{b_1}{t_1} \in \frac{R}{S}$ such that $\frac{a}{s}=\frac{a_1}{s_1}$ and $\frac{b}{t}=\frac{b_1}{t_1}$. Then there exist $u,w \in S$ such that $uas_1 = ua_1s$ and $wbt_1 = wb_1t$. We are going to show that $\frac{a}{s}+\frac{b}{t}=\frac{a_1}{s_1}+\frac{b_1}{t_1}$. We have

$$\begin{aligned} uw(at+bs)s_1t_1 =& (uas_1)(wtt_1)+(ubt_1)(wss_1) = \\ & (ua_1s)(wtt_1)+(ub_1t)(wss_1) = uw(a_1t_1+b_1s_1)st. \end{aligned}$$

Now take $\frac{y}{st} \in \frac{a}{s}+\frac{b}{t}$ with $y \in at+bs$. From the last equation we have that there is $z \in a_1t_1+b_1s_1$ such that $(uw)y(s_1t_1)=(uw)z(st)$, so $\frac{y}{st}=\frac{z}{s_1t_1} \in \frac{a_1}{s_1}+\frac{b_1}{t_1}$. Hence $\frac{a}{s}+\frac{b}{t} \subseteq \frac{a_1}{s_1}+\frac{b_1}{t_1}$. Similarly we show that $\frac{a_1}{s_1}+\frac{b_1}{t_1} \subseteq \frac{a}{s}+\frac{b}{t}$. Thus, the hyperaddition is well-defined. Now we will show that multiplication is well-defined. Since $uas_1 = ua_1s$ and $wbt_1 = wb_1t$, using commutativity of multiplication in R, we have $uwabs_1t_1 = uwa_1b_1st$. Thus, $\frac{a}{s}\cdot\frac{b}{t}=\frac{ab}{st}=\frac{a_1b_1}{s_1t_1}=\frac{a_1}{s_1}\cdot\frac{b_1}{t_1}$, so multiplication is well-defined.

To show associativity we take $\frac{a_1}{s_1}, \frac{a_2}{s_2}, \frac{a_3}{s_3} \in \frac{R}{S}$. We obtain

$$(\frac{a_1}{s_1}+\frac{a_2}{s_2})+\frac{a_3}{s_3} = \{\frac{b}{s_1s_2} \mid b \in a_1s_2+a_2s_1\}+\frac{a_3}{s_3} = \{\frac{y}{s_1s_2s_3} \mid y \in bs_3+a_3s_1s_2,\ \ b \in a_1s_2+a_2s_1\} =$$
$$\{\frac{y}{s_1s_2s_3} \mid y \in (a_1s_2+a_2s_1)s_3+a_3s_1s_2\} = \{\frac{y}{s_1s_2s_3} \mid y \in a_1s_2s_3+(a_2s_3+a_3s_2)s_1\} =$$
$$\{\frac{y}{s_1s_2s_3} \mid y \in a_1s_2s_3+cs_1,\ \ c \in a_2s_3+a_3s_2\} = \frac{a_1}{s_1}+\{\frac{c}{s_2s_3} \mid c \in a_2s_3+a_3s_2\} = \frac{a_1}{s_1}+(\frac{a_2}{s_2}+\frac{a_3}{s_3}),$$

where we used distributivity and associativity in the hyperring R.

Commutativity of the hyperaddition is clear. Observe that the element $\frac{0}{1}$ is a zero in $\frac{R}{S}$. Indeed,

$$\frac{a}{s}+\frac{0}{1} = \{\frac{y}{s} \mid y \in 1a+0s\} = \{\frac{a}{s}\}.$$

Take $\frac{a}{s} \in \frac{R}{S}$ and let $\frac{b}{t} \in \frac{R}{S}$ be such that

$$\frac{0}{1} \in \frac{a}{s}+\frac{b}{t} = \{\frac{y}{s} \mid y \in at+bs\}.$$

This means that there exists $u \in S$ such that $0 = 0su = 1yu = yu$ for $y \in at+bs$. Hence, $0 = yu \in atu+bsu$, so $bsu = -atu$, i.e., $\frac{b}{t}=\frac{-a}{s}$. Thus, the unique inverse of $\frac{a}{s}$ is $\frac{-a}{s} =: -\frac{a}{s}$.

Now we will show that the reversibility axiom holds. Let $\frac{a_3}{s_3} \in \frac{a_1}{s_1} + \frac{a_2}{s_2}$. This means that there is $y \in a_1s_2 + a_2s_1$ such that $\frac{a_3}{s_3} = \frac{y}{s_1s_2}$. Hence there exists $u \in S$ such that $ua_3s_1s_2 = uys_3$. Moreover,

$$ua_3s_1s_2 = uys_3 \in u(a_1s_2 + a_2s_1)s_3 = ua_1s_2s_3 + ua_2s_1s_3.$$

Then

$$ua_2s_1s_3 \in ua_3s_1s_2 - ua_1s_2s_3.$$

Hence, using Lemma 1.3.26, we have

$$\frac{a_2}{s_2} = \frac{ua_2s_1s_3}{us_1s_3s_2} \in \frac{ua_3}{us_3} - \frac{a_1s_2}{s_2s_1} = \frac{a_3}{s_3} - \frac{a_1}{s_1}.$$

This proves that $\frac{R}{S}$ is a canonical hypergroup. Using Remark 1.1.9 we have that $\frac{0}{1}$ is a unique zero in $\frac{R}{S}$, so we can denote $0_{\frac{R}{S}} := \frac{0}{1}$. The commutativity of the multiplication is clear. The multiplicative identity is $\frac{1}{1}$. The only axiom which is left to show is distributivity, i.e.,

$$\frac{a_1}{s_1} \cdot (\frac{a_2}{s_2} + \frac{a_3}{s_3}) = (\frac{a_1}{s_1} \cdot \frac{a_2}{s_2}) + (\frac{a_1}{s_1} \cdot \frac{a_3}{s_3}).$$

We compute

$$\frac{a_1}{s_1} \cdot (\frac{a_2}{s_2} + \frac{a_3}{s_3}) = \frac{a_1}{s_1} \cdot \{\frac{y}{s_2s_3} \mid y \in a_2s_3 + a_3s_2\} = \{\frac{y}{s_1s_2s_3} \mid y \in a_1(a_2s_3 + a_3s_2)\},$$

$$(\frac{a_1}{s_1} \cdot \frac{a_2}{s_2}) + (\frac{a_1}{s_1} \cdot \frac{a_3}{s_3}) = \{\frac{z}{s_1^2s_2s_3} \mid z \in a_1s_1(a_2s_3 + a_3s_2)\} = \{\frac{z}{s_1s_2s_3} \mid z \in a_1(a_2s_3 + a_3s_2)\},$$

where in the last equality we used Lemma 1.3.26. This proves distributivity. □

Definition 1.3.28. The hyperring $\frac{R}{S}$ is called a *hyperring of fractions*.

Definition 1.3.29. A *local hyperring* is a hyperring with a unique maximal hyperideal.

Lemma 1.3.30. *(a) Let S be a multiplicatively closed subset of a hyperring R such that $1 \in S$ and S does not contain zero divisors. The map $\psi : R \to \frac{R}{S}$, $r \mapsto \frac{r}{1}$ is a hyperring monomorphism. The image $\psi(R) =: \frac{R}{1}$ is isomorphic to R.*

(b) A hyperring R is local if and only if $R \setminus R^\times$ is a hyperideal in R. Then $R \setminus R^\times$ is the unique maximal hyperideal in R.

Proof. (a) Consider the mapping $\psi : R \to \frac{R}{S}$ defined as $a \mapsto \frac{a}{1}$. We are going to show that ψ is a monomorphism of hyperrings. From the proof of Proposition 1.3.27 we have that $\psi(0_R) = \frac{0}{1} = 0_{\frac{R}{S}}$. Take $a, b \in R$. Then

$$\psi(a) \cdot \psi(b) = \frac{a}{1} \cdot \frac{b}{1} = \frac{ab}{1} = \psi(ab),$$

$$\psi(a + b) = \{\frac{y}{1} \mid y \in a + b\} = \frac{a}{1} + \frac{b}{1} = \psi(a) + \psi(b).$$

Hence ψ is a strict homomorphism of hyperrings. Take $\frac{a}{1} = \frac{b}{1}$. Then there exists $u \in S$ such that $ua = ub$. Hence $0 \in u(b-a)$, which means that there exists $y \in b-a$ such that $0 = uy$. Since S has no zero divisors and $u \neq 0$, we obtain that $y = 0$, which means that $0 \in b - a$. Hence $a = b$, so ψ is injective. The image $\psi(R) = \frac{R}{1}$ is isomorphic to R. This finishes the proof of part (a).

(b) Let R be a local hyperring with unique maximal hyperideal $\mathcal{M}$. Take any element $a \in R \setminus R^\times$ and consider the set

$$\langle a \rangle := \{ra \mid r \in R\}.$$

This is a proper hyperideal (see Lemma 1.3.17 (a) and Remark 1.3.18). Hence $a \in \langle a \rangle \subseteq \mathcal{M}$ by Lemma 1.3.16, so $R \setminus R^\times \subseteq \mathcal{M}$. The reverse inclusion is clear, since a proper ideal contains only non-units by Lemma 1.3.14. Hence, $\mathcal{M} = R \setminus R^\times$.

Let us assume now that $R \setminus R^\times$ is a hyperideal. By Lemma 1.3.14 we have $I \subseteq R \setminus R^\times$ for every proper hyperideal I. So the hyperideal $R \setminus R^\times$ is the unique maximal in R.

□

Example 1.3.31. (a) Every valuation hyperring is a local hyperring (see Definition 3.1.6 and Lemma 3.1.8).

(b) Let R be a hyperring. If $\mathfrak{p}$ is a prime hyperideal of R and $S := R \setminus \mathfrak{p}$, then $R_\mathfrak{p} := \frac{R}{S}$ is a local hyperring with $\mathfrak{p}R_\mathfrak{p} := \{\frac{p}{s} \mid p \in \mathfrak{p}, s \in R \setminus \mathfrak{p}\}$ being its unique maximal hyperideal (see Theorem 3.6 of [11]).

To prove that, we observe that $S = R \setminus \mathfrak{p}$ is a multiplicatively closed subset with 1. Clearly $1 \notin \mathfrak{p}$, so $1 \in S$. Let $a, b \in S$ and suppose that $ab \notin S$. It means that $ab \in \mathfrak{p}$ and, since $\mathfrak{p}$ is prime, we conclude that $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$, which is a contradiction. Hence $R_\mathfrak{p}$ is a hyperring of fractions by Proposition 1.3.27.

Now we are going to show that $\mathfrak{p}R_\mathfrak{p}$ is a hyperideal in $R_\mathfrak{p}$. Take $\frac{p_1}{s_1}, \frac{p_2}{s_2} \in \mathfrak{p}R_\mathfrak{p}$. Then

$$\frac{p_1}{s_1} + \frac{p_2}{s_2} = \{\frac{y}{s_1 s_2} \mid y \in p_1 s_2 + p_2 s_1\}$$

Since $p_1, p_2 \in \mathfrak{p}$, we obtain that $p_1 s_2, p_2 s_1 \in \mathfrak{p}$ and $p_1 s_2 + p_2 s_1 \subseteq \mathfrak{p}$. Hence $x + y \subseteq \mathfrak{p}R_\mathfrak{p}$ for every $x, y \in pR_\mathfrak{p}$. Take now $\frac{a}{t} \in R_\mathfrak{p}$ and $\frac{p}{s} \in \mathfrak{p}R_\mathfrak{p}$. Then

$$\frac{a}{t} \cdot \frac{p}{s} = \frac{ap}{ts}.$$

Since $p \in \mathfrak{p}$ and $a \in R$, we have $ap \in \mathfrak{p}$. Hence $rx \in \mathfrak{p}R_\mathfrak{p}$ for every $r \in R_\mathfrak{p}$ and $x \in \mathfrak{p}R_\mathfrak{p}$. Moreover, $1_{R_\mathfrak{p}} = \frac{1}{1} \notin \mathfrak{p}R_\mathfrak{p}$, because $1 \notin \mathfrak{p}$. So $\mathfrak{p}R_\mathfrak{p}$ is a proper hyperideal in $R_\mathfrak{p}$. Take $s \in R \setminus \mathfrak{p}$ and observe that $\frac{a}{s} \in R_\mathfrak{p}^\times$ if and only if $a \notin \mathfrak{p}$. Indeed, if $a \notin \mathfrak{p}$, then $a \in R \setminus \mathfrak{p}$, so $\frac{s}{a} \in R_\mathfrak{p}$. Hence, $\frac{a}{s} \in R_\mathfrak{p}^\times$. To prove the converse, assume that $\frac{a}{s} \in R_\mathfrak{p}^\times$. Then $\frac{s}{a} \in R_\mathfrak{p}^\times$. Suppose that $a \in \mathfrak{p}$. Take $t \in R \setminus \mathfrak{p}$ and $\frac{b}{t} \in R_\mathfrak{p}$ such that $\frac{b}{t} = \frac{s}{a}$. Then $uba = uts$ for some $u \in R \setminus \mathfrak{p}$. Since $a \in \mathfrak{p}$ we obtain that $uts \in \mathfrak{p}$. This is a contradiction, since $u, t, s \in R \setminus \mathfrak{p}$ and $\mathfrak{p}$ is a

prime hyperideal. Hence $R_{\mathfrak{p}} \setminus R_{\mathfrak{p}}^{\times} = \mathfrak{p}R_{\mathfrak{p}}$. From Lemma 1.3.30 (b) we obtain that $R_{\mathfrak{p}}$ is a local hyperring with $\mathfrak{p}R_{\mathfrak{p}}$ being its unique maximal hyperideal.

Proposition 1.3.32. *Let R be an integral hyperdomain. Then*

$$R = \bigcap_{\mathcal{M} \in \boldsymbol{M}_R} R_{\mathcal{M}},$$

where $\boldsymbol{M}_R$ is the set of all maximal hyperideals in R and where $R_{\mathcal{M}}$ is the hyperring of fractions constructed as in Example 1.3.31 (b).

Proof. The inclusion $R \subseteq \bigcap_{\mathcal{M} \in \mathbf{M}_R} R_{\mathcal{M}}$ follows from Lemma 1.3.30 (a). To prove the reverse inclusion, suppose that there exists $z \in \bigcap_{\mathcal{M} \in \mathbf{M}_R} R_{\mathcal{M}} \setminus R$. Consider the following set:

$$I = \{x \in R \mid xz \in R\}.$$

We will show that I is a proper hyperideal in R. Take $x, y \in I$. Then $xz \in R$ and $yz \in R$, so $(x -_R y)z = xz -_R yz \subseteq R$. Hence, $x -_R y \subseteq I$. Take $a \in R$. Then $(ax)z = a(xz) \in R$, so $ax \in I$. Moreover, $1 \notin I$, because $z \notin R$. We have shown that I is a proper hyperideal in R. Hence, by Lemma 1.3.16, there exists $\mathcal{M}_0 \in \mathbf{M}_R$ such that $I \subseteq \mathcal{M}_0$. We can write $z = \frac{r}{s} \in R_{\mathcal{M}_0}$ with $r \in R$ and $s \in R \setminus \mathcal{M}_0$. But then $sz = \frac{s}{1} \cdot \frac{r}{s} = \frac{sr}{1s} = \frac{r}{1} = r \in R$ by Lemma 1.3.26, so $s \in I$, which is a contradiction because $I \subseteq \mathcal{M}_0$. Hence, $R = \bigcap_{\mathcal{M} \in \mathbf{M}_R} R_{\mathcal{M}}$. □

Lemma 1.3.33. *Let R be a hyperdomain. Then* $\mathrm{Frac}(R) := \frac{R}{R^*}$ *is a hyperfield.*

Proof. Since R is a hyperdomain, the set R^* contains no zero divisors, so it is a multiplicatively closed subset. From Proposition 1.3.27 we know that $\mathrm{Frac}(R)$ is a hyperring. The only thing that we need to show is that every nonzero element $\frac{a}{b} \in \mathrm{Frac}(R)^*$ has a multiplicative inverse. The inverse is $\frac{b}{a} \in \mathrm{Frac}(R)^*$. Indeed,

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab} = \frac{1}{1} = 1_{\mathrm{Frac}(R)},$$

where for the second equality we used Lemma 1.3.26. □

Definition 1.3.34. Let R be a hyperdomain. Then the hyperfield $\mathrm{Frac}(R) := \frac{R}{R^*}$ is called a *hyperfield of fractions*.

Chapter 2

Real hyperfields

2.1 Artin-Schreier theory for hyperfields

As was mentioned in Example 1.2.7, hyperfields are a generalization of the notion of fields. Hence, it is natural to ask which results from the classical Artin-Schreier theory of ordered fields (for reference see [2], [22], [28], [12], [41]) are still valid for the theory of hyperfields. That first approach was done by M. Marshall in [34]. In the following, we are going to recall terminology and results from that paper.

Definition 2.1.1. Let F be a hyperfield. A subset $P \subseteq F$ is called a *positive cone* if

$$P + P \subseteq P, \qquad P \cdot P \subseteq P, \qquad P \cap -P = \emptyset, \qquad P \cup -P = F^*.$$

A hyperfield is called a *real hyperfield* if it admits at least one positive cone.

Example 2.1.2. Every real field can be turned into a real hyperfield by setting $x + y := \{x + y\}$.

Example 2.1.3. Consider the sign hyperfield $\mathbb{S} = \mathbb{R}_{(\mathbb{R}^*)^2}$. It admits the positive cone $P_{(\mathbb{R}^*)^2} = \{1\}$.

Note that every real field is infinite. Indeed, suppose that K is a finite field and admits a positive cone P. Since $\operatorname{char} K = p$ and $1 \in P$ we obtain that

$$-1 = \underbrace{1 + \ldots + 1}_{p-1 \text{ times}} \in P$$

and this is a contradiction. However, as we have observed in the last example, there exist finite real hyperfields. Let us prove the following general property of finite real hyperfields, which we will use later.

Lemma 2.1.4. *Let (F, P) be a finite real hyperfield. Then*

$$\bigcup_{x \in P} 1 + x = P.$$

Proof. Suppose that there exists $a \in P$ such that $a \notin 1 + x$ for any $x \in P$. If $a = 1$, then $(1 - 1) \cap P = \emptyset$. Since the set $1 - 1$ is symmetric, we obtain that $1 - 1 = \{0\}$. The finite real hyperfield F is not a field, so we have a contradiction to Lemma 1.2.4.

Since F^* is a finite group, every element in this group has a finite order. Let $\mathrm{ord}(a) = n$. The assumption that $a \notin 1 + x$ for any $x \in P$ implies that $(a - 1) \cap P = \emptyset$. So $1 - a \subseteq P$ and this implies that $a^{-1} - 1 \subseteq P$. We observe:

$$a^{-1} - a \subseteq a^{-1} - 1 + 1 - a \subseteq P.$$

Multiplying the last line by a we obtain $1 - a^2 \subseteq P$. Hence $a^{-2} - 1 \subseteq P$. Furthermore,

$$a^{-2} - a = a^{-2} - 1 + 1 - a \subseteq P.$$

Multiplying the last line by a^2 we have $1 - a^3 \subseteq P$. Proceeding like this, after n steps, we obtain that $1 - a^n = 1 - 1 \subseteq P$, which is a contradiction. □

Definition 2.1.5. Let F be a hyperfield. A subset $T \subseteq F^*$ is called a *prepositive cone* if

$$T + T \subseteq T, \qquad T \cdot T \subseteq T, \qquad (F^*)^2 \subseteq T.$$

Proposition 2.1.6. *Let F be a hyperfield and T a prepositive cone in F. Then:*
(a) $-1 \notin T$,
(b) T is a subgroup of F^*,
(c) $T \cap -T = \emptyset$.

Proof. To prove (a) observe that if $-1 \in T$, then $0 \in 1 + (-1) \subseteq T$, which is a contradiction to $T \subseteq F^*$. To prove (b) we observe first that $1 = 1^2 \in T$. Moreover, if $a \in T$, then $a^{-1} = a \cdot (a^{-1})^2 \in T$. To prove (c) assume that there exists $a \in T$ such that $a \in T \cap -T$. Then $a \in T$ and $-a \in T$ and $0 \in -a + a \subseteq T$, which is a contradiction to $T \subseteq F^*$. □

A prepositive cone T is called *maximal* if for any prepositive cone S we have that $T \subseteq S$ implies $T = S$. We will denote by $\mathcal{X}(F)$ the set of all positive cones in a hyperfield F and by $\mathcal{X}(F \mid T)$ the set of all positive cones in a hyperfield F which contain T.

Example 2.1.7. Let F be a hyperfield such that $-1 \notin \sum (F^*)^2$. Then $\sum (F^*)^2$ is a prepositive cone in F. Indeed, $\sum (F^*)^2$ is additively and multiplicatively closed and contains the set of nonzero squares. If $0 \in \sum (F^*)^2$, then $0 \in a_1^2 + ... + a_n^2$ for $a_1, ..., a_n \neq 0$. This implies that

$$-a_1^2 \in a_2^2 + ... + a_n^2$$

and since $a_1 \neq 0$,

$$-1 \in (\frac{a_2}{a_1})^2 + ... + (\frac{a_n}{a_1})^2 \in \sum (F^*)^2,$$

which is a contradiction to our assumption. Moreover, $\sum(F^*)^2$ is contained in every positive cone $P \in \mathcal{X}(F)$ (because positive cones are additively and multiplicatively closed).

Remark 2.1.8. In his paper Marshall dealt with structures called multifields. However, although multirings and hyperrings are slightly different structures, it was mentioned in [16] that multifields and hyperfields are the same thing.

The following results have been proved by Marshall in [34] and later extended in [16] by Marshall and Gładki. However, the results in these papers are stated for positive cones and prepositive cones defined as sets with 0 included (cf. the set P_0 in (4.1) in Section 4.2 of this thesis). Hence, we give alternative proofs of these results, using notions from Definition 2.1.1 and Definition 2.1.5.

Lemma 2.1.9 (Corollary 3.3, [16])**.** *Let F be a hyperfield and T a prepositive cone. Then $F = T - T$.*

Proof. Clearly $T - T \subseteq F$. To see the other inclusion, take any $x \in F$.

Assume first that $x = -1$. Take any $y \in T$. Then $y + 1 \subseteq T$. Observe that $-1 \in y - (y + 1)$, so there is $w \in y + 1 \subseteq T$ such that $-1 \in y - w \subseteq T - T$.

Now assume that $x \neq -1$ and take $z \in x + 1$. Then $z \neq 0$ and

$$z^2 \in (x + 1)^2 \subseteq x^2 + 1 + x(1 + 1).$$

This means that there are $s \in x^2 + 1 \subseteq T$ and $t \in 1 + 1 \subseteq T$ such that $z^2 \in s + xt$. That, by the reversibility axiom, is equivalent to $xt \in z^2 - s$. We assumed $t \in T$, so $t \neq 0$. Hence $x \in \frac{z^2}{t} - \frac{s}{t} \subseteq T - T$. □

Lemma 2.1.10. *Let T be a prepositive cone in a hyperfield F. Let $a \in F^*$ be an element such that $-a \notin T$. Then the set $T + aT := \bigcup_{s,t \in T}(s + at)$ is also a prepositive cone in F. Moreover, $T \subseteq T + aT$ and $a \in T + aT$.*

Proof. If $0 \in T + aT$, then there are $s, t \in T$ such that $0 \in s + at$. Hence $-a = \frac{s}{t} \in T$, which is a contradiction to the assumption. Hence $T + aT \subseteq F^*$. Take $s_1, s_2, t_1, t_2 \in T$. We obtain

$$(s_1 + at_1) + (s_2 + at_2) = (s_1 + s_2) + a(t_1 + t_2) \subseteq T + aT,$$

$$(s_1 + at_1) \cdot (s_2 + at_2) \subseteq (s_1 s_2 + a^2 t_1 t_2) + a(t_1 s_2 + s_1 t_2) \subseteq T + aT.$$

From Lemma 2.1.9 we know that $a \in s - t$ for some $s, t \in T$. Then $s \in t + a$. Since $s \in T \subseteq F^*$, we have $1 \in \frac{t}{s} + \frac{a}{s}$. Take any $u \in T$. Multiplying the latter by u, we obtain $u \in \frac{t}{s} \cdot u + a \cdot \frac{u}{s} \subseteq T + aT$. Hence $T \subseteq T + aT$ and since $(F^*)^2 \subseteq T$, we have $(F^*)^2 \subseteq T \subseteq T + aT$. This proves that $T + aT$ is a prepositive cone. From Lemma 2.1.9 we know that $\frac{1}{a} \in s - t$ for some $s, t \in T$. Then $1 \in as - at$. From the reversibility axiom, $as \in 1 + at$. Multiplying the latter by $\frac{1}{s}$, we have $a \in \frac{1}{s} + a \cdot \frac{t}{s} \subseteq T + aT$. This finishes the proof. □

Proposition 2.1.11 (Lemma 3.3, [34]). *Let F be a real hyperfield. A maximal prepositive cone in F is a positive cone in F.*

Proof. Let F be a hyperfield and T a maximal prepositive cone in F. We will show that T is a positive cone. Clearly, T is additively and multiplicatively closed. We have already proved in Proposition 2.1.6 that $T \cap -T = \emptyset$. It is left to show that $T \cup -T = F^*$. Let $a \in F^*$ be such that $-a \notin T$. By Lemma 2.1.10 we know that $T + aT$ is a prepositive cone in F such that $T \subseteq T + aT$. Since T is assumed to be a maximal prepositive cone, we obtain that $T = T + aT$. It follows from Lemma 2.1.10 that $a \in T + aT = T$. This completes the proof. □

Corollary 2.1.12. *Every prepositive cone T in a hyperfield F is contained in some positive cone $P \in \mathcal{X}(F)$.*

Proof. Consider a chain $(T_i)_{i \in I}$ of prepositive cones containing a prepositive cone T. The union $\bigcup_{i \in I} T_i$ is again a prepositive cone. Indeed, if $a, b \in \bigcup_{i \in I} T_i$, then if follows from the chain property that for some $j \in I$ we have $a, b \in T_j$. Then $a + b \subseteq T_j$ and $ab \in T_j$. Moreover, $\bigcup_{i \in I} T_i \subseteq F^*$ and $(F^*)^2 \subseteq T \subseteq \bigcup_{i \in I} T_i$.

From Zorn's Lemma, there exists a maximal prepositive cone in F, which, by Proposition 2.1.11, is a positive cone. □

A classical variant of the following theorem is one of the fundamental results in the theory of ordered fields. It links the existence of an order relation in a field to the purely algebraic properties of that field. It was proved by E. Artin and O. Schreier. The analog for hyperfields was proved by Marshall.

Theorem 2.1.13 (Lemma 3.3, [34]). *A hyperfield F is real if and only if $-1 \notin \sum (F^*)^2$.*

Proof. Suppose that $-1 \in \sum (F^*)^2$ and that F admits a positive cone P. Since $\sum (F^*)^2 \subseteq P$ for every P, we obtain $-1 \in P$. Then $1 \in P \cap -P$, which is a contradiction.

Assume now that $-1 \notin \sum (F^*)^2$. Then $T_0 := \sum (F^*)^2$ is a prepositive cone, as we have proved in Example 2.1.7. From Corollary 2.1.12 we know that there exists a positive cone P which contains T_0. □

Example 2.1.14. Consider the Krasner hyperfield $\mathbb{K}$ from Example 1.2.9. Since $0 \in 1 + 1$, we have that $-1 = 1$. Thus, $-1 \in \sum (F^*)^2$, so $\mathbb{K}$ is not real.

The following lemma will be useful later:

Lemma 2.1.15. *Let F be a real hyperfield. Then*

$$0 \in a_1^2 + ... + a_n^2 \Leftrightarrow a_1 = ... = a_n = 0.$$

Proof. If $a_1 = ... = a_n = 0$, then $a_1^2 + ... + a_n^2 = \{0\}$. To see the other inclusion, suppose that $0 \in a_1^2 + ... + a_n^2$ and not all a_i are zero. Without loss of generality we can assume that $a_1 \neq 0$. Then

$$-a_1^2 \in a_2^2 + ... + a_n^2$$

and since $a_1 \neq 0$,

$$-1 \in (\frac{a_2}{a_1})^2 + ... + (\frac{a_n}{a_1})^2 \in \sum (F^*)^2,$$

which is a contradiction to Theorem 2.1.13. □

Theorem 2.1.16 (Proposition 3.4, [34])**.** *Let F be a real hyperfield. For any prepositive cone $T \subseteq F$,*

$$T = \bigcap_{P \in \mathcal{X}(F|T)} P.$$

Proof. A prepositive cone T is clearly contained in the intersection of positive cones $P \in \mathcal{X}(F \mid T)$. To prove the reverse inclusion, assume that $a \notin T$. We have to show that there exists a positive cone $P \in \mathcal{X}(F \mid T)$ such that $a \notin P$. By Lemma 2.1.10, the set $T - aT$ is a prepositive cone such that $T \subseteq T - aT$ and $-a \in T - aT$. By Corollary 2.1.12 there exists a positive cone P such that $T - aT \subseteq P$. Hence $-a \in P$ and therefore $a \notin P$. □

Corollary 2.1.17. *Let F be a real hyperfield. The set $\sum (F^*)^2$ is the intersection of all positive cones $P \in \mathcal{X}(F)$.*

Proof. Observe that $\sum (F^*)^2$ is a prepositive cone contained in every positive cone P by Example 2.1.7. Using Theorem 2.1.16 we obtain the statement. □

Proposition 2.1.18 (Remark 3.2, [26])**.** *A hyperfield F is real if and only if there exists a nontrivial homomorphism of F into the sign hyperfield $\mathbb{S}$.*

Proof. Let P be a positive cone of F. Consider the map $\phi : F \to \mathbb{S}$ defined as follows:

$$\phi(a) = \begin{cases} 1 & \text{if } a \in P, \\ 0 & \text{if } a = 0, \\ -1 & \text{if } a \in -P. \end{cases}$$

This is a nontrivial homomorphism of hyperfields. To prove the converse, assume that we have a nontrivial homomorphism of hyperfields $\phi : F \to \mathbb{S}$. Suppose that F is not real. Then $-1 \in \sum (F^*)^2$ by Theorem 2.1.13. Observe that from axioms (HH2) and (HH3) of Definition 1.3.4, we have

$$\phi(-1) \in \phi(a_1^2 + \ldots + a_n^2) \subseteq \phi(a_1^2) + \ldots + \phi(a_n^2) = \{1\}$$

for some $a_1, \ldots, a_n \in F^*$. Hence, $\phi(-1) = 1$ and this is a contradiction to Proposition 1.3.6. Thus, F is a real hyperfield. □

Let F be a hyperfield and $A \subseteq F$ a subset of F. We define the set

$$\langle A \rangle := \{c \in F \mid c \in \sum_{i=1}^{n} a_i, n \in \mathbb{N}, a_i \in \prod A\},$$

where $\prod A$ is the set of all finite products of elements from A.

The next lemma has been proved for real fields in [27]. We state and prove an analogous result for hyperfields.

Lemma 2.1.19. *Let F be a real hyperfield and $A \subseteq F$. The following conditions are equivalent:*

(a) there exists $P \in \mathcal{X}(F)$ such that $A \subseteq P$,

(b) $0 \notin \langle A \cup (F^*)^2 \rangle$,

(c) $\langle A \cup (F^*)^2 \rangle$ is a prepositive cone in F.

Proof. To show that (a) implies (b) assume that $A \subseteq P$ for some $P \in \mathcal{X}(F)$. Then $A \cup (F^*)^2 \subseteq P$, because $(F^*)^2$ is contained in every positive cone in F. Since P is additively and multiplicatively closed, $\langle A \cup (F^*)^2 \rangle \subseteq P$. Hence, $0 \notin \langle A \cup (F^*)^2 \rangle$. To show that (b) implies (c), assume that $0 \notin \langle A \cup (F^*)^2 \rangle$. Observe that $\langle A \cup (F^*)^2 \rangle$ is additively and multiplicatively closed and $(F^*)^2 \subseteq \langle A \cup (F^*)^2 \rangle$. Hence, $\langle A \cup (F^*)^2 \rangle$ is a prepositive cone in F. The fact that (c) implies (a) follows from Corollary 2.1.12. □

The following lemma is a classical result in topology, stated e.g. in Section 3 in [44].

Lemma 2.1.20 (Alexander's Compactness Theorem)**.** *If a topological space X has a subbasis S of X such that from every cover of X by elements of S, a finite subcover can be selected, then X is compact.*

One can introduce a topology on $\mathcal{X}(F)$ by taking a subbasis given by the sets

$$H(a) = \{P \in \mathcal{X}(F) \mid a \in P\}.$$

Moreover, such topology has the following, interesting properties.

Theorem 2.1.21 (Proposition 3.1, [34])**.** *Let F be a hyperfield. Then $\mathcal{X}(F)$ is a Boolean topological space, i.e., a totally disconnected, compact Hausdorff space.*

Proof. Observe that $H(0) = \emptyset$ and for every $a \in F^*$ the open set $H(a)$ has the complement $H(-a)$, which also is open. Hence every open set is clopen, and this shows that $\mathcal{X}(F)$ is totally disconnected. Moreover, take two distinct positive cones $P_1, P_2 \in \mathcal{X}(F)$ and any $a \in P_1 \setminus P_2$. Then $P_1 \in H(a)$ and $P_2 \in H(-a)$ and $H(a) \cap H(-a) = \emptyset$. Thus $\mathcal{X}(F)$ is a Hausdorff space. It is left to show that $\mathcal{X}(F)$ is compact. To this end, we will use Lemma 2.1.20. Consider a cover of $\mathcal{X}(F)$ by elements

of the subbasis, i.e., $\mathcal{X}(F) = \bigcup_{a\in A} H(-a)$ for some $A \subseteq F^*$. Then $\bigcap_{a\in A} H(a) = \emptyset$, so there is no positive cone $P \in \mathcal{X}(F)$ such that $A \subseteq P$. From Lemma 2.1.19 we obtain that $0 \in \langle A \cup (F^*)^2\rangle$. Hence, we have that $0 \in a_1x_1 + \ldots + a_nx_n + x_{n+1} + \ldots + x_k$ for some $a_i \in \prod A, i \in \{1, \ldots, n\}$ and $x_j \in (F^*)^2, j \in \{1, \ldots, k\}, k \geq n$. Let $B := \{a_1, \ldots, a_n\}$. Then $0 \in \langle B \cup (F^*)^2\rangle$, so by Lemma 2.1.19, there is no positive cone $P \in \mathcal{X}(F)$ such that $B \subseteq P$. Thus, $\bigcap_{a\in B} H(a) = \emptyset$, so $\mathcal{X}(F) = \bigcup_{a\in B} H(-a)$. Since B is finite, we have found a finite subcover of $\mathcal{X}(F)$ given by the subbasis sets. □

The topology on $\mathcal{X}(F)$ described above is called the *Harrison topology*.

2.2 Real factor hyperfields

In this section we restrict our attention to real factor hyperfields. Factor hyperfields are introduced in Theorem 1.2.5. We will prove a criterion for a factor hyperfield K_T to be real.

Lemma 2.2.1 (Lemma 3.3, [26]). *Let K be a real field with a positive cone P. Let T be a multiplicative subgroup of K^* such that $T \subseteq P$. Consider the factor hyperfield $K_T = \{[a]_T \mid a \in K\}$.*

(a) We have $a \in P$ if and only if $[a]_T \subseteq P$.

(b) The set $P_T := \{[a]_T \mid a \in P\}$ is a positive cone in K_T.

(c) Let Q be a positive cone in K_T. Then the set

$$\widetilde{Q} = \bigcup_{[c]_T \in Q} [c]_T$$

is a positive cone in K such that $T \subseteq \widetilde{Q}$.

Proof. (a) Since $T \subseteq P$, we obtain that $at \in P$ for every $a \in P$ and every $t \in T$. Hence $[a]_T = \{at \mid t \in T\} \subseteq P$. The reverse implication is trivial.

(b) Take $a, b \in P$. Then

$$[a]_T + [b]_T = \{[as + bt]_T \mid s, t \in T\}.$$

Since $T \subseteq P$, we have that $s, t \in P$ for every $s, t \in T$. Hence $as + bt \in P$ and $[a]_T + [b]_T \subseteq P_T$. The fact that $[a]_T \cdot [b]_T \in P_T$ follows from the fact that $ab \in P$. So the axioms $P_T + P_T \subseteq P_T$ and $P_T \cdot P_T \subseteq P_T$ are shown. Assume that there is $[a]_T \in K_T$ such that $[a]_T \in P_T \cap -P_T$. Then $a \in P$ and $-a \in P$, which is a contradiction. Hence $P_T \cap -P_T = \emptyset$. Finally, let $[0]_T \neq [a]_T \notin P_T$. This means that $a \notin P$, hence $-a \in P$ and $-[a]_T = [-a]_T \in P$. This proves that $P_T \cup -P_T = (K_T)^*$.

(c) Clearly $\widetilde{Q} \subseteq K$. We have $T = [1]_T \in Q$, so $T \subseteq \widetilde{Q}$. Pick $a, b \in \widetilde{Q}$. Then $[a]_T, [b]_T \in Q$ and since Q is a positive cone in K_T, we have $[a + b]_T \in [a]_T + [b]_T \subseteq Q$, so $a + b \in \widetilde{Q}$. Moreover,

$[ab]_T = [a]_T \cdot [b]_T \in Q$, so $ab \in \widetilde{Q}$. This shows that $\widetilde{Q} + \widetilde{Q} \subseteq \widetilde{Q}$ and $\widetilde{Q} \cdot \widetilde{Q} \subseteq \widetilde{Q}$. Take $a \in K^*$. If $[a]_T \in Q$, then $a \in \widetilde{Q}$. If $-[a]_T = [-a]_T \in Q$, then $-a \in \widetilde{Q}$, so $a \in -\widetilde{Q}$. This proves $\widetilde{Q} \cup -\widetilde{Q} = K^*$. Finally, assume that $a \in \widetilde{Q} \cap -\widetilde{Q}$. Then there exist $[c]_T \in Q$ such that $a \in [c]_T$ and $[d]_T \in -Q$ such that $a \in [d]_T$. From the construction of a factor hyperfield, we obtain that $[a]_T = [c]_T$ and $[a]_T = [d]_T$. This means that $[a]_T \in Q \cap -Q$, which is a contradiction. Hence $\widetilde{Q} \cap -\widetilde{Q} = \emptyset$. This proves that $\widetilde{Q}$ is a positive cone in K and $T \subseteq \widetilde{Q}$. □

Theorem 2.2.2. *Let K be a real field and K_T a factor hyperfield, where $T \subseteq K^*$. Then the functions*

$$\Psi : \mathcal{X}(K \mid T) \to \mathcal{X}(K_T) \qquad \Psi^{-1} : \mathcal{X}(K_T) \to \mathcal{X}(K \mid T)$$
$$P \mapsto P_T \qquad Q \mapsto \widetilde{Q}$$

are bijections and inverses of each other, where

$$P_T = \{[c]_T \mid c \in P\} \quad \text{and} \quad \widetilde{Q} = \bigcup_{[c]_T \in Q} [c]_T.$$

Proof. If $P \in \mathcal{X}(K \mid T)$, then $P_T \in \mathcal{X}(K_T)$ by Lemma 2.2.1 (b). We are going to show that Ψ is a bijection. To prove that this map is injective, take $P, Q \in \mathcal{X}(K \mid T)$ such that $P \neq Q$ and let $a \in P \setminus Q$. Then $-a \in Q$, so $-[a]_T = [-a]_T \in Q_T$. On the other hand, $[a]_T \in P_T$. Therefore, $P_T \neq Q_T$.

To prove that Ψ is surjective, take $Q \in \mathcal{X}(K_T)$. We claim that $\Psi(\widetilde{Q}) = Q$, where

$$\widetilde{Q} = \bigcup_{[c]_T \in Q} [c]_T.$$

By Lemma 2.2.1 (c) we know that $\widetilde{Q} \in \mathcal{X}(K \mid T)$. Using the fact that $[ct]_T = [c]_T$ for any $t \in T$ and for any $c \in K$, we compute

$$\Psi(\widetilde{Q}) = \{[c]_T \mid c \in \widetilde{Q}\} = \{[c]_T \mid c \in \bigcup_{[d]_T \in Q} [d]_T\} = \{[c]_T \mid c = dt, [d]_T \in Q, t \in T\} = \{[c]_T \mid [c]_T \in Q\} = Q.$$

Hence Ψ is a bijection. It is left to show that Ψ^{-1} is the inverse function to Ψ. Assume that $P \in \mathcal{X}(K \mid T)$. Then

$$\Psi^{-1}(\Psi(P)) = \Psi^{-1}(P_T) = \bigcup_{[c]_T \in P_T} [c]_T = \bigcup_{c \in P} [c]_T = \{ct \mid c \in P, t \in T\} \subseteq P,$$

where in the last inclusion we use the fact that $T \subseteq P$ and P is multiplicatively closed. Moreover, using the fact that $1 \in T$, for every $c \in P$ we have also that $c = c \cdot 1 \in \{ct \mid c \in P, t \in T\} = \Psi^{-1}(\Psi(P))$, so $\Psi^{-1}(\Psi(P)) = P$. Finally, since the inverse function to a bijection is a bijection, we conclude that Ψ^{-1} is also a bijection. □

From the last theorem we conclude the following.

Corollary 2.2.3 (Theorem 3.4, [26]). *Let K be a field and T a subgroup of K^*. Then*

$$|\mathcal{X}(K_T)| = |\mathcal{X}(K|T)|.$$

In particular, $\mathcal{X}(K_T) \neq \emptyset$ if and only if $\mathcal{X}(K|T) \neq \emptyset$.

Example 2.2.4. Consider the sign hyperfield $\mathbb{S} = \mathbb{R}_{(\mathbb{R}^*)^2}$. We have seen in Example 2.1.3 that $\mathbb{S}$ is a real hyperfield with the positive cone $P_{(\mathbb{R}^*)^2} = \{1\}$. Since the set $(\mathbb{R}^*)^2$ is the unique positive cone in $\mathbb{R}$, from Corollary 2.2.3 we have that $P_{(\mathbb{R}^*)^2} = \{1\}$ is a unique positive cone in $\mathbb{S}$.

Example 2.2.5. Consider the factor hyperfield $F := \mathbb{Q}_{(\mathbb{Q}^*)^2}$, where $(\mathbb{Q}^*)^2$ denotes the set of nonzero squares of rational numbers. The field of rational numbers is real and it admits a unique positive cone $\mathbb{Q}^+$ associated with its natural ordering. Since the set $(\mathbb{Q}^*)^2$ is contained in $\mathbb{Q}^+$, we obtain that F is real with a unique positive cone $P_{(\mathbb{Q}^*)^2} = \{[a]_{(\mathbb{Q}^*)^2} \mid a \in \mathbb{Q}^+\}$.

In the next example we use facts and constructions derived from the theory of real closed fields and its connection to valuation theory. Let us give a brief survey on the results which we will use.

Let K be a real field and take $P \in \mathcal{X}(K)$. For $a \in K$ we define

$$|a| := \begin{cases} a & \text{if } a \in P \cup \{0_K\} \\ -a & \text{if } a \in -P. \end{cases}$$

We say that two elements $a, b \in K$ are *archimedean equivalent* if there exists $n \in \mathbb{N}$ such that $n|a| - |b| \in P$ and $n|b| - |a| \in P$. One can check that archimedean equivalence is an equivalence relation. We denote by $\tilde{a}$ the archimedean class of $a \in K$. Define an ordering $<$ on the set $\Gamma := \{\tilde{a} \mid a \in K\}$ as follows:

$$\tilde{a} < \tilde{b} \iff |a| - |b| \in P \text{ and } \tilde{a} \neq \tilde{b}. \tag{2.1}$$

With this ordering and the operation $\tilde{a} + \tilde{b} := \widetilde{ab}$ we have that Γ is an ordered abelian group. The function $v_P : K \to \Gamma \cup \{\infty\}$ defined as $a \mapsto \tilde{a}$ if $a \neq 0_K$ and $0_K \mapsto \infty$ is a valuation on the ordered field K and is called the *natural valuation* induced by P. It is a nontrivial valuation if and only if K is non-archimedean ordered with respect to P, i.e., there exist $a, b \in K^*$ such that $|a| - n|b| \in P$ for every $n \in \mathbb{N}$. In such a situation we say that a is *archimedean larger* than b (and b is *archimedean smaller* than a). Note that different positive cones in K may induce the same natural valuation.

A *real closed field* is an ordered field which has no proper algebraic extension that is real.

For a general reference on orderings, valuations and real closed fields, see e.g. Section 2 of [28] and [22].

Example 2.2.6 ([26], Example 3.6)**.** Consider a non-archimedean ordered real closed field R with its unique positive cone $P := (R^*)^2$. Let $<$ be the ordering associated with P and let v be the

natural valuation associated with P. Denote by $A(P)$ the valuation ring associated with v and by $I(P)$ its unique maximal ideal. Let $E^+(R) := A(P)^\times \cap P$ be the set of positive units of $A(P)$. Then $E^+(R)$ is a subgroup of R^*. Consider the factor hyperfield $F := R_{E^+(R)}$, which we will call the *hyperfield of pairs*. Observe that for $a, b \in R^*$ we have

$$[a]_{E^+(R)} = [b]_{E^+(R)} \Leftrightarrow \frac{a}{b} \in E^+(R) \Leftrightarrow v\big(\frac{a}{b}\big) = 0 \text{ and } \operatorname{sgn}\big(\frac{a}{b}\big) = 1 \Leftrightarrow v(a) = v(b) \text{ and } \operatorname{sgn}(a) = \operatorname{sgn}(b),$$

where $\operatorname{sgn}(a)$ is the sign of a in the unique ordering of R. Hence, we can identify nonzero elements of F as

$$[a]_{E^+(R)} = (\operatorname{sgn}(a), v(a)).$$

Since $E^+(R)$ is a subgroup of positive elements of R, we obtain by Corollary 2.2.3 that F is a real hyperfield and has a unique positive cone. Moreover, by Theorem 2.2.2 we obtain that $P_{E^+(R)} = \{(1, \gamma) \mid \gamma \in \Gamma\}$ is precisely this positive cone.

Using the definition of multiplication in factor hyperfields, for elements from F^* we obtain

$$(\operatorname{sgn}(a), v(a)) \cdot (\operatorname{sgn}(b), v(b)) = (\operatorname{sgn}(ab), v(ab)) = (\operatorname{sgn}(a) \cdot \operatorname{sgn}(b), v(a) + v(b)).$$

Moreover, multiplication by $[0]_{E^+(R)}$ results in $[0]_{E^+(R)}$. The element $(1, 0)$ is the neutral element for multiplication and the multiplicative inverse for $(e, \gamma) \in F^*$ is $(e, -\gamma)$.

Take $a, b \in R$. Then

$$[a]_{E^+(R)} + [b]_{E^+(R)} = \{[as + bt]_{E^+(R)} \mid s, t \in E^+(R)\}.$$

We are going to show that the results of hyperaddition are the following:

$$\begin{aligned}
(1, \gamma_1) + (1, \gamma_2) &= \{(1, \min\{\gamma_1, \gamma_2\})\} && \gamma_1, \gamma_2 \in \Gamma \\
(-1, \gamma_1) + (-1, \gamma_2) &= \{(-1, \min\{\gamma_1, \gamma_2\})\} && \gamma_1, \gamma_2 \in \Gamma \\
(1, \gamma_1) + (-1, \gamma_2) &= \{(1, \gamma_1)\} && \gamma_1 < \gamma_2 \in \Gamma \\
(1, \gamma_1) + (-1, \gamma_2) &= \{(-1, \gamma_2)\} && \gamma_1 > \gamma_2 \in \Gamma \\
(1, \gamma) + (-1, \gamma) &= \{(e, \delta) \mid e \in \{-1, 1\}, \delta \geq \gamma\} \cup \{[0]_{E^(R)}\} && \gamma \in \Gamma
\end{aligned} \tag{2.2}$$

To see the first equality, assume that $\operatorname{sgn}(a) = \operatorname{sgn}(b) = 1$ and let $v(a) = \gamma_1$ and $v(b) = \gamma_2$. We obtain $\operatorname{sgn}(as + bt) = 1$ for all $s, t \in E^+(R)$ since $\operatorname{sgn}(s) = \operatorname{sgn}(t) = 1$. Obviously, $v(s) = v(t) = 0$. We use the following property of the natural valuation on a field (see e.g. Theorem 2.3 in [28]):

$$\operatorname{sgn}(a) = \operatorname{sgn}(b) \Rightarrow v(a + b) = \min\{v(a), v(b)\}. \tag{2.3}$$

Hence, we obtain $v(as + bt) = \min\{v(as), v(bt)\} = \min\{v(a), v(b)\} = \min\{\gamma_1, \gamma_2\}$, which proves the first equality. The proof of the second equality is similar. To prove the third equality, take $a, b \in R$ such that $\operatorname{sgn}(a) = 1$, $\operatorname{sgn}(b) = -1$ and $\gamma_1 = v(a) < v(b) = \gamma_2$. Take any $s, t \in E^+(R)$.

Then $v(as) = v(a) < v(b) = v(bt)$, hence by (2.1), $as + bt = as - (-bt) = |as| - |bt| \in P$. Hence, $\operatorname{sgn}(as + bt) = 1$ and $v(as + bt) = \min\{v(as), v(bt)\} = v(as) = v(a) = \gamma_1$. The proof of the fourth equality is similar.

Take now $a \in R$ such that $v(a) = \gamma$. Then

$$[a]_{E^+(R)} + [-a]_{E^+(R)} = \{[as - at]_{E^+(R)} \mid s, t \in E^+(R)\}.$$

We have $v(as - at) = v(a) + v(s - t) \geq v(a) + \min\{v(s), v(t)\} = v(a) = \gamma$. Hence $[a]_{E^+(R)} + [-a]_{E^+(R)} \subseteq \{(e, \delta) \mid e \in \{-1, 1\}, \delta \geq \gamma\} \cup \{[0]_{E^+(R)}\}$. To prove the reverse inclusion, take $\delta \geq \gamma$. Since v is surjective, there exists some $b \in R$ such that $\delta = v(b)$. Moreover, we can choose b in such way that $ba^{-1} > 0$ (if $ba^{-1} < 0$, then we take $-b$ instead of b). We have $v(ba^{-1}) = \delta - \gamma \geq 0$ and hence $1 + ba^{-1} \in E^+(R)$. Taking $s = 1 + ba^{-1}$ and $t = 1$, we obtain $v(as - at) = v(b) = \delta$. Note that $\operatorname{sgn}(as - at) = -\operatorname{sgn}(at - as)$, so the set is symmetric around $[0]_{E^+(R)}$. This proves the reverse inclusion and finishes the proof of the equalities in (2.2).

The additive inverse of the element $(e, \gamma) \in F^*$ is $(-e, \gamma)$. Indeed, $(e, \gamma) + (-e, \gamma) = \{(e, \delta) \mid e \in \{-1, 1\}, \delta \geq \gamma\} \cup \{[0]_{E^(R)}\} \ni [0]_{E^(R)}$.

Consider the subset $S := \{(-1, 0), [0]_{E^+(R)}, (1, 0)\} \subseteq F$. For $a, b \in S$, we have that $ab \in S$ and $(S, +_S, \cdot, [0]_{E^+(R)})$ is a hyperring, where $a +_S b := (a + b) \cap S$. Thus, S is a subhyperring of F. Note that S is not a strict subhyperring of F. Indeed, since R is non-archimedean ordered we have that $I(P) \neq \{0\}$. Thus, there exists an element $a \in P \setminus E^+(R)$ such that $v(a) = \delta > 0$. So $[a]_{E^+(R)} \in P_{E^+(R)}$, $[a]_{E^+(R)} \neq (1, 0)$ and $[a]_{E^+(R)} \in (1, 0) + (-1, 0)$. Hence $S \subsetneq (1, 0) + (-1, 0)$.

Finally, let $\mathbb{S}$ be the sign hyperfield. Consider the map $\varphi : \mathbb{S} \to S$ defined as follows: $1 \mapsto (1, 0), -1 \mapsto (-1, 0)$ and $0 \mapsto [0]_{E^+(R)}$. Since φ is a strict, bijective hyperring homomorphism, we can identify the set S with the sign hyperfield. Thus, the sign hyperfield is a subhyperfield of F.

The hyperfield of pairs, described in the last example, is a stringent hyperfield. In fact, we have a more general result about the real stringent hyperfields. Recall that by Theorem 1.2.16, every stringent hyperfield is of the form $\mathbb{H} \rtimes G$, where $\mathbb{H}$ is either the Krasner hyperfield $\mathbb{K}$ or the sign hyperfield $\mathbb{S}$ or a field.

Proposition 2.2.7 (Proposition 3.13, [36]). *A stringent hyperfield $F = \mathbb{H} \rtimes G$ is real if and only if $\mathbb{H} = \mathbb{S}$ or $\mathbb{H}$ is a real field.*

Proof. We will show that $F = \mathbb{H} \rtimes G$ is real if and only if $\mathbb{H}$ is real. Assume that $\mathbb{H}$ is real with a positive cone P. We are going to prove that the set $P_F := \{(x, g) \mid x \in P\}$ is a positive cone in F. Take $(x_1, g_1), (x_2, g_2) \in P_F$. If $g_1 > g_2$, then $(x_1, g_1) + (x_2, g_2) = \{(x_1, g_1)\} \subseteq P_F$. Analogously, if $g_2 > g_1$, then $(x_1, g_1) + (x_2, g_2) = \{(x_2, g_2)\} \subseteq P_F$. Observe that the situation $0_{\mathbb{H}} \in x_1 +_{\mathbb{H}} x_2$ is impossible, since $x_1 +_{\mathbb{H}} x_2 \subseteq P$. Hence, if $g_1 = g_2$, we obtain $(x_1, g_1) + (x_2, g_2) = \{(y, g) \mid z \in x_1 +_{\mathbb{H}} x_2\} \subseteq P_F$, since $x_1 +_{\mathbb{H}} x_2 \subseteq P$. Moreover, $(x_1, g_1) \cdot (x_2, g_2) = (x_1 \cdot_{\mathbb{H}} x_2,\ g_1 +_G g_2) \in P_F$, because $x_1 \cdot_{\mathbb{H}} x_2 \in P$.

The facts that $P_F \cap -P_F = \emptyset$ and $P_F \cup -P_F = F^*$ follow from the same properties of P. Hence, P_F is a positive cone in F, i.e., F is real. To prove the converse, assume that F is a real hyperfield. By Proposition 2.1.18 there exists a nontrivial homomorphism of hyperfields $\phi : F \to \mathbb{S}$, where $\mathbb{S}$ is the sign hyperfield. Define a hyperfield homomorphism $\varphi : \mathbb{H} \to F$ given by

$$\varphi(a) = \begin{cases} (a, 0), & \text{if } a \neq 0_{\mathbb{H}}, \\ 0, & \text{if } a = 0_{\mathbb{H}}. \end{cases}$$

Then φ is injective and the composition $\phi \circ \varphi$ is a nontrivial homomorphism of $\mathbb{H}$ into $\mathbb{S}$. Thus, from Proposition 2.1.18, it is a real hyperfield.

From Theorem 1.2.16 we know that every stringent hyperfield is of the form $\mathbb{H} \rtimes G$, where $\mathbb{H} = \mathbb{K}$ or $\mathbb{H} = \mathbb{S}$ or $\mathbb{H}$ is a field. From Example 2.2.4 and Example 2.1.14 we know that $\mathbb{S}$ is a real hyperfield, while $\mathbb{K}$ is not. Hence, we conclude that the only possibilities for $\mathbb{H}$ to be real are $\mathbb{H} = \mathbb{S}$ or $\mathbb{H}$ to be a real field. This completes the proof. □

One could think that every real hyperfield is a factor hyperfield. Let us present an example of a real hyperfield which is not a factor hyperfield. First, we will need the following theorem which was proved by V. Bergelson and B. Shapiro in [5] and by D. Turnwald in [45].

Theorem 2.2.8. *If K is an infinite field and G is a subgroup of K^* of finite index, then $G - G = K$.*

Now we will use the idea of M. Baker and T. Jin from [3] to prove that the following hyperfield is not a factor hyperfield.

Example 2.2.9 ([20], Example 4)**.** Consider the set $F = \{-a^2, -a, -1, 0, 1, a, a^2\}$ and its subset $P := \{1, a, a^2\}$. We define on F the following hyperaddition:

$+$	$-a^2$	$-a$	-1	0	1	a	a^2
$-a^2$	$\{-1, -a\}$	$-P$	$-P$	$\{-a^2\}$	F^*	F^*	$F \setminus \{a^2, -a^2\}$
$-a$	$-P$	$\{-1, -a^2\}$	$-P$	$\{-a\}$	F^*	$F \setminus \{a, -a\}$	F^*
-1	$-P$	$-P$	$\{-a, -a^2\}$	$\{-1\}$	$F \setminus \{1, -1\}$	F^*	F^*
0	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$
1	F^*	F^*	$F \setminus \{1, -1\}$	$\{1\}$	$\{a, a^2\}$	P	P
a	F^*	$F \setminus \{a, -a\}$	F^*	$\{a\}$	P	$\{1, a^2\}$	P
a^2	$F \setminus \{a^2, -a^2\}$	F^*	F^*	$\{a^2\}$	P	P	$\{1, a\}$

and multiplication:

$\cdot$	$-a^2$	$-a$	-1	0	1	a	a^2
$-a^2$	a	1	a^2	0	$-a^2$	-1	$-a$
$-a$	1	a^2	a	0	$-a$	$-a^2$	-1
-1	a^2	a	1	0	-1	$-a$	$-a^2$
0	0	0	0	0	0	0	0
1	$-a^2$	$-a$	-1	0	1	a	a^2
a	-1	$-a^2$	$-a$	0	a	a^2	1
a^2	$-a$	-1	$-a^2$	0	a^2	1	a

Then $A_1 := (F, +, \cdot, 0, 1)$ is a hyperfield. Moreover, F is real since P is its positive cone. Suppose that A_1 is a factor hyperfield for some field K and a multiplicative subgroup T in K^*. From Corollary 2.2.3 we have that K admits a positive cone. Hence, K is a real field, i.e., it is infinite. Moreover, the index of T in F^* is 6, hence finite. Thus, from Theorem 2.2.8 we have that $T - T = K$, so in particular, $1 \in 1 - 1$. This is a contradiction.

It is natural to ask whether there exists also an infinite real hyperfield which is not a factor hyperfield. Note that all examples of infinite hyperfields we have already seen are factor hyperfields. Unfortunately, at the moment this question remains open.

Chapter 3

Valued hyperfields

The first hyperfields considered by M. Krasner in [24] were of a very particular form. In fact they were factor hyperfields $K_{1+\mathcal{M}^\gamma}$, where K is a field with valuation v and $\mathcal{M}^\gamma = \{a \in K \mid v(a) > \gamma\}$, where γ is an initial segment in the value group $v(K^*)$. These hyperfields inherit a valuation from K. Hence, the first hyperfields were valued hyperfields. However, the definition of a valued hyperfield given in [24] (and later repeated in [29]) is very restrictive. In [11] B. Davvaz and A. Salasi introduced a more general definition of a valuation on a hyperring. We will use this definition for our results.

The valuation in the sense proposed by Davvaz and Salasi is a map from a hyperring (or a hyperfield) into an ordered abelian group. In [37] K. H. Mirdar Harijani and S. Anvariyeh were dealing with a hypervaluation, which is a map from a hyperfield onto an ordered canonical hypergroup. However, it was proved in [30] that every hypervaluation in the sense introduced by Mirdar Harijani and Anvariyeh can be decomposed into an order preserving homomorphism of hypergroups and a valuation in the sense proposed by Davvaz and Salasi, which induces the same valuation hyperring.

3.1 Valuations and valuation hyperrings

The following definition was given originally for hyperrings as Definition 4.2 in [11]. We use it to define a valuation on a hyperfield:

Definition 3.1.1 (Definition 4.1, [26])**.** Let F be a hyperfield and let Γ be an ordered abelian group (written additively). A surjective map $v : F \to \Gamma \cup \{\infty\}$ is called a *valuation* if it satisfies the following properties for all $x, y, z \in F$:

(V1) $v(x) = \infty \Leftrightarrow x = 0$,

(V2) $v(xy) = v(x) + v(y)$,

(V3) $z \in x + y \Rightarrow v(z) \geq \min\{v(x), v(y)\}$.

Lemma 3.1.2 (Lemma 4.3, [11], Lemma 4.5, [26]). *Let $v : F \to \Gamma \cup \{\infty\}$ be a valuation on a hyperfield F. Then for every $x, y, z \in F$:*

(a) $v(1) = v(-1) = 0$,

(b) $v(x) = v(-x)$,

(c) $v(x^{-1}) = -v(x)$,

(d) if $v(x) \neq v(y)$ and $z \in x + y$, then $v(z) = \min\{v(x), v(y)\}$.

Proof. (a) From axiom (V2), $v(1) = v(1 \cdot 1) = v(1) + v(1)$, so $v(1) = 0$. Moreover, $0 = v(1) = v((-1) \cdot (-1)) = v(-1) + v(-1)$, which implies that also $v(-1) = 0$.

(b) For any $x \in F$, we have $v(-x) = v(-1) + v(x) = v(x)$, where we used part (a) and axiom (V2).

(c) For any $x \in F^*$, $0 = v(1) = v(x \cdot x^{-1}) = v(x) + v(x^{-1})$, where we used part (a) and axiom (V2). Hence $v(x^{-1}) = -v(x)$.

(d) Without loss of generality, assume $v(x) < v(y)$ and let $z \in x + y$. By axiom (V3) we obtain that $v(z) \geq \min\{v(x), v(y)\} = v(x)$. From the reversibility axiom, $x \in z - y$, so

$$v(x) \geq \min\{v(z), v(-y)\} = \min\{v(z), v(y)\} \geq v(x),$$

where we used part (b) and axiom (V3). Hence, $v(x) = \min\{v(z), v(y)\}$. Since $v(x) \neq v(y)$, we conclude that $v(z) = v(x)$.

□

Let F be a hyperfield and $v : F \to \Gamma \cup \{\infty\}$ a valuation on F. Then we denote the value group by $v(F^*)$. If $v(F^*) = \{0\}$, we say that v is the *trivial valuation*.

Lemma 3.1.3. *Every finite hyperfield admits only the trivial valuation.*

Proof. If a hyperfield is finite and admits a valuation v, then the value group $v(F^*)$ is also finite. The only finite ordered abelian group is the trivial one. □

Example 3.1.4. The sign hyperfield $\mathbb{S}$, the Krasner hyperfield $\mathbb{K}$ and the hyperfield A_1 from Example 2.2.9 admit only trivial valuations.

Example 3.1.5. Another example of trivially valued hyperfields are Massouros hyperfields. Let v be a valuation on a Massouros hyperfield F. Since $1 + 1 = F \setminus \{1\}$, we obtain from axiom (V3) and Lemma 3.1.2 (a) that $v(x) \geq 0$ for every $x \in F$. Assume that there exists $x \in F$ such that $v(x) > 0$. Then $v(x^{-1}) < 0$ by Lemma 3.1.2 (c), which is a contradiction. Hence v is the trivial valuation.

The results in this section follow from [26]. However, for the reader's convenience, we will recall the proofs.

Definition 3.1.6 (Definition 4.6, [26]). Let F be a hyperfield. A subhyperring $\mathcal{O}$ of F is called a *valuation hyperring* if for every $x \in F$ we have $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

Proposition 3.1.7 (Proposition 4.7, [26]). *A valuation hyperring of a hyperfield F is a strict subhyperring in F.*

Proof. Take $x, y \in \mathcal{O}$. We need to show that $x - y \subseteq \mathcal{O}$. Take $z \in x - y$ and suppose that $z \notin \mathcal{O}$. Then $z^{-1} \in \mathcal{O}$ and thus $xz^{-1}, yz^{-1} \in \mathcal{O}$. Since $z \in x - y$, from the reversibility axiom, we obtain $x \in z + y$. Now, using axiom (R3), we have

$$xz^{-1} \in (z + y)z^{-1} = 1 + yz^{-1}.$$

Hence,

$$xz^{-1} \in (1 + yz^{-1}) \cap \mathcal{O} = 1 +_{\mathcal{O}} yz^{-1}.$$

Again, by axiom (C4) we obtain

$$1 \in xz^{-1} -_{\mathcal{O}} yz^{-1}.$$

Using axiom (R3) we have

$$z \in x -_{\mathcal{O}} y = (x - y) \cap \mathcal{O},$$

which, in particular, means that $z \in \mathcal{O}$. □

Lemma 3.1.8 (Lemma 4.8, [26]). *Let $\mathcal{O}$ be a valuation hyperring in a hyperfield F. Then $\mathcal{O}$ is a local hyperring with $\mathcal{M} := \mathcal{O} \setminus \mathcal{O}^{\times}$ being its unique maximal hyperideal.*

Proof. We are going to show that $\mathcal{M}$ is a hyperideal in F. Take $r \in \mathcal{O}$ and $a \in \mathcal{M}$. If $ra \notin \mathcal{M}$, then $ra \in \mathcal{O}^{\times}$, so there exists $s \in \mathcal{O}$ such that $1 = s(ra) = (sr)a$ with $sr \in \mathcal{O}$. Hence a is invertible, which is a contradiction.

Take $a, b \in \mathcal{M}$. We may assume that $ab^{-1} \in \mathcal{O}$ (otherwise $ba^{-1} \in \mathcal{O}$). Then $1 - ab^{-1} \subseteq \mathcal{O}$, since by Proposition 3.1.7, $\mathcal{O}$ is a strict subhyperring in F. Thus,

$$b - a = b(1 - ab^{-1}) \subseteq \mathcal{M},$$

by the first part of the proof. By Lemma 1.3.30 (b), $\mathcal{O}$ is a local hyperring.

□

Definition 3.1.9 (Definition 4.9, [26]). Let F be a hyperfield, $\mathcal{O}$ a valuation hyperring in F and $\mathcal{M}$ its unique maximal hyperideal. The quotient hyperring $\overline{F} := \mathcal{O}/\mathcal{M}$ is, by Proposition 1.3.24, a hyperfield. It is called the *residue hyperfield*.

Proposition 3.1.10 (Proposition 4.11, [26]). *Let v be a valuation on a hyperfield F. Then $\mathcal{O}_v := \{x \in F \mid v(x) \geq 0\}$ is a valuation hyperring of F and $\mathcal{M}_v := \{x \in F \mid v(x) > 0\}$ is the unique maximal hyperideal in $\mathcal{O}_v$.*

Proof. First we will show that $\mathcal{O}_v$ is a subhyperring in F. Take $x, y \in \mathcal{O}$ and consider any $z \in x + y$. Using axiom (V3), we obtain $v(z) \geq \min\{v(x), v(y)\} \geq 0$, which shows that $x + y \subseteq \mathcal{O}_v$. Moreover, by axiom (V2), we have $v(xy) = v(x) + v(y) \geq 0$. Hence $xy \in \mathcal{O}_v$.

Now we show that $\mathcal{O}_v$ is a valuation hyperring. Take $x \in F$. If $x \notin \mathcal{O}_v$, then $v(x) < 0$. Hence, by Lemma 3.1.2 (c), $v(x^{-1}) > 0$. Thus $x^{-1} \in \mathcal{O}_v$.

Observe that $\mathcal{M}_v = \mathcal{O}_v \setminus \mathcal{O}_v^\times$. Thus, by Lemma 3.1.8, the set $\mathcal{M}_v$ is the unique maximal hyperideal in $\mathcal{O}_v$. □

Proposition 3.1.11 (Proposition 4.12, [26]). *Let F be a hyperfield and $\mathcal{O}$ a valuation hyperring in F. Let $\mathcal{O}^\times$ be the group of units of $\mathcal{O}$. Consider the group $\Gamma := F^\times/\mathcal{O}^\times$ and define a relation $\leq$ on Γ as follows*

$$a\mathcal{O}^\times \leq b\mathcal{O}^\times \iff ba^{-1} \in \mathcal{O}.$$

Then $(\Gamma, \cdot, \leq)$ is an ordered abelian group and the canonical projection $\pi : F \to \Gamma \cup \{\infty\}$, extended so that $\pi(0) = \infty$, is a valuation on F. Moreover, $\mathcal{O} = \mathcal{O}_\pi$.

Proof. We will start with showing that $\leq$ is well-defined. Take $a_1\mathcal{O}^\times = a_2\mathcal{O}^\times, b_1\mathcal{O}^\times = b_2\mathcal{O}^\times$ and assume that $a_1\mathcal{O}^\times \leq b_1\mathcal{O}^\times$. We wish to show that $a_2\mathcal{O}^\times \leq b_2\mathcal{O}^\times$. We have that $b_2b_1^{-1}, a_1a_2^{-1} \in \mathcal{O}^\times$ and $b_1a_1^{-1} \in \mathcal{O}$. Then

$$b_2a_2^{-1} = b_2b_1^{-1} \cdot b_1a_1^{-1} \cdot a_1a_2^{-1} \in \mathcal{O}.$$

Thus, $a_2\mathcal{O}^\times \leq b_2\mathcal{O}^\times$. Now we will prove that $\leq$ is a non-strict linear order. Since $aa^{-1} = 1 \in \mathcal{O}$, we obtain that $a\mathcal{O}^\times \leq a\mathcal{O}^\times$, so the relation is reflexive. To show antisymmetry, take $a\mathcal{O}^\times, b\mathcal{O}^\times \in \Gamma$ such that $a\mathcal{O}^\times \leq b\mathcal{O}^\times$ and $b\mathcal{O}^\times \leq a\mathcal{O}^\times$. Then $ba^{-1} \in \mathcal{O}$ and $(ba^{-1})^{-1} = ab^{-1} \in \mathcal{O}$, hence $ba^{-1} \in \mathcal{O}^\times$. So $a\mathcal{O}^\times = b\mathcal{O}^\times$. Take now $a\mathcal{O}^\times, b\mathcal{O}^\times, c\mathcal{O}^\times \in \Gamma$ with $a\mathcal{O}^\times \leq b\mathcal{O}^\times$ and $b\mathcal{O}^\times \leq c\mathcal{O}^\times$. Then $ba^{-1}, cb^{-1} \in \mathcal{O}$. We compute $ca^{-1} = (cb^{-1}) \cdot (ba^{-1}) \in \mathcal{O}$, which implies that $a\mathcal{O}^\times \leq c\mathcal{O}^\times$. Thus $\leq$ is transitive. Moreover, the relation is total, because $ab^{-1} \in \mathcal{O}$ or $ba^{-1} \in \mathcal{O}$ for every $a, b \in F^*$.

We will prove that the relation $\leq$ is compatible with the group operation. Take $a\mathcal{O}^\times \leq b\mathcal{O}^\times$ and $c\mathcal{O}^\times \in \Gamma$. Then $ba^{-1} \in \mathcal{O}$ and $ba^{-1} = (bc)(ac)^{-1}$. Thus $ac\mathcal{O}^\times \leq bc\mathcal{O}^\times$.

Finally, we will show that π is a valuation. The map π is a surjection onto $\Gamma \cup \{\infty\}$ and the axiom (V1) clearly holds. Moreover, $\pi(a) \cdot \pi(b) = a\mathcal{O}^\times \cdot b\mathcal{O}^\times = ab\mathcal{O}^\times = \pi(ab)$, so the axiom (V2) also holds. To show the axiom (V3), take $c \in a + b$ and assume that $\pi(a) \leq \pi(b)$. If $b = 0$, then clearly $\pi(c) = \pi(a)$. If $b \neq 0$, then also $a \neq 0$ and $ba^{-1} \in \mathcal{O}$. Hence

$$ca^{-1} \in (a + b)a^{-1} = 1 + ba^{-1} \subseteq \mathcal{O},$$

where in the last inclusion we used the fact that $\mathcal{O}$ is a strict subhyperring, which follows from Proposition 3.1.7. We obtain that $\pi(c) \geq \pi(a)$, so the map π is a valuation.

To prove the last assertion, observe that

$$x \in \mathcal{O}_\pi \Leftrightarrow x\mathcal{O}^\times \geq 1\mathcal{O}^\times \Leftrightarrow x \in \mathcal{O}.$$

This proves that $\mathcal{O}_\pi = \mathcal{O}$. □

We finish this subsection with two lemmas which will be useful later.

Lemma 3.1.12 (Lemma 4.10, [26]). *Take a hyperfield F and two valuation hyperrings $\mathcal{O}_1$, $\mathcal{O}_2$ in F. Let $\mathcal{M}_i$ be the maximal hyperideal of $\mathcal{O}_i$ for $i = 1, 2$. Then*

$$\mathcal{O}_1 \subseteq \mathcal{O}_2 \Leftrightarrow \mathcal{M}_2 \subseteq \mathcal{M}_1.$$

Proof. Let v_i be the valuation associated with $\mathcal{O}_i$ for $i = 1, 2$. We have $F = \mathcal{O}_i \dot{\cup} (\mathcal{M}_i^\times)^{-1}$, where $(\mathcal{M}^\times)^{-1} := \{x \in F^* \mid v_i(x) < 0\}$. Thus, $(\mathcal{M}_i^\times)^{-1} = F \setminus \mathcal{O}_i$ for $i = 1, 2$. If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then $F \setminus \mathcal{O}_2 \subseteq F \setminus \mathcal{O}_1$. Hence $(\mathcal{M}_2^\times)^{-1} \subseteq (\mathcal{M}_1^\times)^{-1}$. For $x \in F^*$ and using Lemma 3.1.2 (c), we have

$$\begin{aligned}(x \in (\mathcal{M}_2^\times)^{-1} \Rightarrow x \in (\mathcal{M}_1^\times)^{-1}) &\Leftrightarrow (v_2(x) < 0 \Rightarrow v_1(x) < 0)\\ \Leftrightarrow (v_2(x^{-1}) > 0 \Rightarrow v_1(x^{-1}) > 0) &\Leftrightarrow (x^{-1} \in \mathcal{M}_2 \Rightarrow x^{-1} \in \mathcal{M}_1).\end{aligned}$$

The latter is equivalent to $\mathcal{M}_2 \subseteq \mathcal{M}_1$. □

Lemma 3.1.13. *Let F be a hyperfield and $\mathcal{O}_1$ a valuation hyperring of F. If $\mathcal{O}_1 \subseteq \mathcal{O}_2$ and $\mathcal{O}_2$ is a subhyperring in F, then $\mathcal{O}_2$ is also a valuation hyperring.*

Proof. Take any $x \in F$ and let $\mathcal{O}_2$ be a subhyperring in F. Then $x \in \mathcal{O}_1$ or $x^{-1} \in \mathcal{O}_1$. Since $\mathcal{O}_1 \subseteq \mathcal{O}_2$, it is true that $x \in \mathcal{O}_2$ or $x^{-1} \in \mathcal{O}_2$. Hence $\mathcal{O}_2$ is a valuation hyperring in F. □

3.2 Valued factor hyperfields

Now we are going to investigate valued factor hyperfields.

Lemma 3.2.1. *Let K be a field with a valuation ring $\mathcal{O}$. Let K_T be a factor hyperfield, where $T \subseteq K^*$ is a multiplicative subgroup in K and $T \subseteq \mathcal{O}^\times$.*

(a) We have $c \in \mathcal{O}$ if and only if $[c]_T \subseteq \mathcal{O}$.

(b) The set $\mathcal{O}_T := \{[c]_T \in K_T \mid c \in \mathcal{O}\}$ is a valuation hyperring in K_T.

(c) Let $\mathcal{A}$ be a valuation hyperring in K_T. Then the set $\widetilde{\mathcal{A}} := \{c \in K \mid [c]_T \in \mathcal{A}\}$ is a valuation ring in K and $T \subseteq \widetilde{\mathcal{A}}^\times$.

Proof. (a) Since $T \subseteq \mathcal{O}^\times$, we obtain that $ct \in \mathcal{O}$ for every $c \in \mathcal{O}$ and every $t \in T$. Hence $[c]_T = \{ct \mid t \in T\} \subseteq \mathcal{O}$. The reverse implication is trivial.

(b) We will show first that $\mathcal{O}_T$ is a strict subhyperring in K_T. Take $[a]_T, [b]_T \in \mathcal{O}_T$. Then $[a]_T - [b]_T = \{[as - bt]_T \mid s, t \in T\}$. Observe that $at - bs \in \mathcal{O}$ for every $s, t \in T$, since $a, b \in \mathcal{O}$ by part (a). Hence, $[a]_T - [b]_T \subseteq \mathcal{O}_T$. Moreover, $[a]_T \cdot [b]_T = [ab]_T \in \mathcal{O}_T$, because $ab \in \mathcal{O}$. Hence, $\mathcal{O}_T$ is a strict subhyperring in K_T. Take now $[c]_T \in K_T$. If $c \in \mathcal{O}$, then $[c]_T \in \mathcal{O}_T$. If $c \notin \mathcal{O}$, then $c^{-1} \in \mathcal{O}$, because $\mathcal{O}$ is a valuation ring. Hence ${[c]_T}^{-1} = [c^{-1}]_T \in \mathcal{O}_T$, so $\mathcal{O}_T$ is a valuation hyperring in K_T.

(c) We will show first that $\widetilde{\mathcal{A}}$ is a subring in K. Take $a, b \in \widetilde{\mathcal{A}}$. Then $[a - b]_T \in [a]_T - [b]_T \subseteq \mathcal{A}$, so $a - b \in \widetilde{\mathcal{A}}$. Moreover, $[a]_T \cdot [b]_T = [ab]_T \in \mathcal{A}$, so $ab \in \widetilde{\mathcal{A}}$. Hence, $\widetilde{\mathcal{A}}$ is a subring in K. Take $c \in K$. If $[c]_T \in \mathcal{A}$, then $c \in \widetilde{\mathcal{A}}$. If $[c]_T \notin \mathcal{A}$, then $[c^{-1}]_T = {[c]_T}^{-1} \in \mathcal{A}$, because $\mathcal{A}$ is a valuation hyperring. Hence $c^{-1} \in \widetilde{\mathcal{A}}$, so $\widetilde{\mathcal{A}}$ is a valuation ring. Moreover, take any $t \in T$. Then also $t^{-1} \in T$. Hence $[t]_T = [t^{-1}]_T = [1]_T \in \mathcal{A}$, so $t \in \widetilde{\mathcal{A}}^\times$. This proves that $T \subseteq \widetilde{\mathcal{A}}^\times$. □

Theorem 3.2.2. *Let K be a field and K_T a factor hyperfield, where $T \subseteq K^\times$ is a multiplicative subgroup in K. Denote by $\mathcal{V}_{K|T}$ the set of all valuation rings $\mathcal{O}$ in K such that $T \subseteq \mathcal{O}^\times$ and by $\mathcal{V}_{K_T}$ the set of all valuation hyperrings in the factor hyperfield K_T. Then the functions:*

$$\begin{aligned} \varphi : \mathcal{V}_{K|T} &\to \mathcal{V}_{K_T} & \qquad \varphi^{-1} : \mathcal{V}_{K_T} &\to \mathcal{V}_{K|T} \\ \mathcal{O} &\mapsto \mathcal{O}_T & \mathcal{A} &\mapsto \widetilde{A} \end{aligned}$$

are bijections and inverses of each other, where

$$\mathcal{O}_T = \{[c]_T \in K_T \mid c \in \mathcal{O}\} \qquad \text{and} \qquad \widetilde{\mathcal{A}} = \{c \in K \mid [c]_T \in \mathcal{A}\}.$$

Proof. If $\mathcal{O} \in \mathcal{V}_{K|T}$, then $\mathcal{O}_T \in \mathcal{V}_{K_T}$ by Lemma 3.2.1 (b). We are going to show that the map φ is a bijection. First we will show that φ is injective. Take $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{V}_{K|T}$ such that $\mathcal{O}_1 \neq \mathcal{O}_2$. Then, without loss of generality, we can assume that there is $c \in \mathcal{O}_2 \setminus \mathcal{O}_1$. Hence $[c]_T \in \mathcal{O}_{T2}$, but $[c]_T \notin \mathcal{O}_{T1}$, because of Lemma 3.2.1 (a). Thus, $\varphi(\mathcal{O}_1) = \mathcal{O}_{T1} \neq \mathcal{O}_{T2} = \varphi(\mathcal{O}_2)$. Now we will show that φ is surjective. Take $\mathcal{O}_{K_T} \in \mathcal{V}_{K_T}$. Let

$$\widetilde{\mathcal{O}_{K_T}} = \{c \in K \mid [c]_T \in \mathcal{O}_{K_T}\}.$$

From Lemma 3.2.1 (c) we know that $\widetilde{\mathcal{O}_{K_T}} \in \mathcal{V}_{K|T}$. We compute

$$\varphi(\widetilde{\mathcal{O}_{K_T}}) = \{[c]_T \in K_T \mid c \in \widetilde{\mathcal{O}_{K_T}}\} = \{[c]_T \in K_T \mid [c]_T \in \mathcal{O}_{K_T}\} = \mathcal{O}_{K_T}.$$

Hence φ is surjective, so it is a bijection. It is left to show that φ^{-1} is the inverse function to φ. Assume that $\mathcal{O} \in \mathcal{V}_{K|T}$. Then

$$\varphi^{-1}(\varphi(\mathcal{O})) = \varphi^{-1}(\mathcal{O}_T) = \{c \in K \mid [c]_T \in \mathcal{O}_T\} = \{c \in K \mid c \in \mathcal{O}\} = \mathcal{O},$$

where in the third equality we use Lemma 3.2.1 (a). Moreover, since the inverse function to a bijection is a bijection, we conclude that φ^{-1} is bijective. □

From Proposition 3.1.11 we know that to any valuation hyperring one can associate a valuation. Let T be a multiplicative subgroup of K^*. Assume that there exists a valuation ring $\mathcal{O}$ such that $T \subseteq \mathcal{O}^\times$ and let $v : K \to v(K^*) \cup \{\infty\}$ be a valuation on K associated to $\mathcal{O}$. Consider the map v_T defined as follows:

$$\begin{aligned} v_T : K_T &\to v(K^*) \cup \{\infty\} \\ [a]_T &\mapsto v(a). \end{aligned}$$

Proposition 3.2.3. *The map v_T is a valuation on K_T and $\mathcal{O}_T = \{[c]_T \in K_T \mid c \in \mathcal{O}\}$ as defined in Theorem 3.2.2 is its valuation hyperring.*

Proof. First we will show that v_T is well-defined. Take $a, b \in K$ such that $[a]_T = [b]_T$. Then $a = bt$ for some $t \in T$. Since $T \subseteq \mathcal{O}^\times$, we have $v(t) = 0$. We compute

$$v_T([a]_T) = v(a) = v(bt) = v(b) + v(t) = v(b) = v_T([b]_T).$$

Now we are going to check whether v_T fulfills the axioms (V1)−(V3) from Definition 3.1.1. First, observe that $v_T([x]_T) = \infty$ is equivalent to $v(x) = \infty$. This holds if and only if $x = 0$. The latter is equivalent to $[x]_T = [0]_T$. This shows that v_T fulfills axiom (V1). Next, take $[x]_T, [y]_T \in K_T$. We have

$$v_T([xy]_T) = v(xy) = v(x) + v(y) = v_T([x]_T) + v_T([y]_T),$$

which proves axiom (V2). Finally, let $[z]_T \in [x]_T + [y]_T$. Then there exist $s, t \in T$ such that $z = xs + yt$. Since $s, t \in T \subseteq \mathcal{O}_v^\times$, we have $v(s) = v(t) = 0$ and

$$v_T([z]_T) = v_T([xs+yt]_T) = v(xs+vt) \geq \min\{v(xs), v(yt)\} = \min\{v(x), v(y)\} = \min\{v_T([x]_T), v_T([y]_T)\}.$$

This proves that v_T is a valuation. Moreover,

$$\mathcal{O}_{v_T} = \{[c]_T \in K_T \mid v_T([c]_T) \geq 0\} = \{[c]_T \in K_T \mid v(c) \geq 0\} = \{[c]_T \in K_T \mid c \in \mathcal{O}\} = \mathcal{O}_T,$$

so v_T is associated with the valuation hyperring $\mathcal{O}_T$. □

Example 3.2.4 (Example 4.4, [26])**.** Consider the hyperfield of pairs $F = R_{E^+(R)}$ introduced in Example 2.2.6 with v being the natural valuation associated with the unique positive cone $(R^*)^2$ in R. Since $E^+(R) \subseteq \mathcal{O}_v$, from Proposition 3.2.3 we obtain that F admits a valuation defined as follows:

$$\begin{aligned} v_{E^+(R)} : F &\to \Gamma \cup \{\infty\} \\ (s, \gamma) &\mapsto \gamma \\ [0]_{E^+(R)} &\mapsto \infty, \end{aligned}$$

where Γ is the value group of the natural valuation in R associated with its unique positive cone.

Now we would like to see what the valuation associated with the valuation ring $\widetilde{\mathcal{A}}$ looks like. Let w be a valuation on K_T associated with a valuation hyperring $\mathcal{A}$. Consider the following map $\widetilde{w}$:

$$\begin{aligned} \widetilde{w} : K &\to w(K_T^*) \cup \{\infty\} \\ x &\mapsto w([x]_T) \end{aligned}$$

Proposition 3.2.5. *The map $\widetilde{w}$ is a valuation on K and $\widetilde{\mathcal{A}} = \{c \in K \mid [c]_T \in \mathcal{A}\}$ as defined in Theorem 3.2.2 is its valuation ring.*

Proof. Since every field is a hyperfield, we can check whether axioms (V1)−(V3) from Definition 3.1.1 are fulfilled. First observe that $\widetilde{w}(c) = \infty$ is equivalent to $w([c]_T) = \infty$. The latter holds if and only if $[c]_T = [0]_T$, which is equivalent to $c = 0$. Hence axiom (V1) holds. To see that axiom (V2) is fulfilled, we compute

$$\widetilde{w}(bc) = w([bc]_T) = w([b]_T) + w([c]_T) = \widetilde{w}(b) + \widetilde{w}(c).$$

Finally, we observe that if $c = a + b$, then $[c]_T = [a + b]_T \in [a]_T + [b]_T$. Hence

$$\widetilde{w}(c) = w([c]_T) \geq \min\{w([a]_T), w([b]_T)\} = \min\{\widetilde{w}(a), \widetilde{w}(b)\},$$

so axiom (V3) holds. This proves that $\widetilde{w}$ is a valuation on K. Moreover,

$$\mathcal{O}_{\widetilde{w}} = \{c \in K \mid \widetilde{w}(c) \geq 0\} = \{c \in K \mid w([c]_T) \geq 0\} = \{c \in K \mid [c]_T \in \mathcal{A}\} = \widetilde{\mathcal{A}}.$$

□

Definition 3.2.6. Let F be a hyperfield and let $v_1 : F \to \Gamma_1 \cup \{\infty\}$ and $v_2 : F \to \Gamma_2 \cup \{\infty\}$ be valuations on F. We say that v_1 and v_2 are *equivalent* if $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$.

Corollary 3.2.7. *Let K be a field and K_T a factor hyperfield, where $T \subseteq K^*$ is a multiplicative subgroup in K. Denote by $\mathcal{W}_{K|T}$ the set of all valuations in K such that T is a subset of the groups of units of their valuation rings and by $\mathcal{W}_{K_T}$ the set of all valuations in the factor hyperfield K_T. Then the functions:*

$$\begin{aligned} \Phi : \mathcal{W}_{K|T} &\to \mathcal{W}_{K_T} \\ v &\mapsto v_T \end{aligned} \qquad \begin{aligned} \Phi^{-1} : \mathcal{W}_{K_T} &\to \mathcal{W}_{K|T} \\ w &\mapsto \widetilde{w}. \end{aligned}$$

are bijections and inverses of each other, where

$$\begin{aligned} v_T : K_T &\to v(K^*) \cup \{\infty\} \\ [x]_T &\mapsto v(x) \end{aligned} \qquad \begin{aligned} \widetilde{w} : K &\to w(K_T{}^*) \cup \{\infty\} \\ x &\mapsto w([x]_T) \end{aligned}$$

for every $x \in K$.

Proof. The proof follows from Proposition 3.2.3, Proposition 3.2.5 and Theorem 3.2.2. □

The following, easy lemma will be useful later.

Lemma 3.2.8. *Let v be a valuation on a field K with a valuation ring $\mathcal{O}_v$ and a maximal ideal $\mathcal{M}_v$ and let w be a valuation on a factor hyperfield K_T with a valuation hyperring $\mathcal{O}_w$ and a maximal hyperideal $\mathcal{M}_w$. Let v_T and $\widetilde{w}$ be valuations as described in Corollary 3.2.7. Then*

$$\mathcal{M}_{v_T} = \{[c]_T \in K_T \mid c \in \mathcal{M}_v\} \qquad \text{and} \qquad \mathcal{M}_{\widetilde{w}} = \{c \in K \mid [c]_T \in \mathcal{M}_w\}.$$

Proof. From Proposition 3.2.3 we know that $\mathcal{O}_{v_T} = \{[c]_T \in K_T \mid c \in \mathcal{O}_v\}$. Observe that $\mathcal{O}_{v_T}^\times = \{[c]_T \in K_T \mid c \in \mathcal{O}_v^\times\}$. Indeed, $[c]_T \in \mathcal{O}_{v_T}^\times$ if and only if $[c]_T \in \mathcal{O}_{v_T}$ and $[c^{-1}]_T \in \mathcal{O}_{v_T}$. This is equivalent to $c \in \mathcal{O}_v$ and $c^{-1} \in \mathcal{O}_v$. The latter holds if and only if $c \in \mathcal{O}_v^\times$. Now from Lemma 3.1.8 we have that

$$\mathcal{M}_{v_T} = \mathcal{O}_{v_T} \setminus \mathcal{O}_{v_T}^\times = \{[c]_T \in K_T \mid c \in \mathcal{O}_v \setminus \mathcal{O}_v^\times\} = \{[c]_T \in K_T \mid c \in \mathcal{M}_v\}.$$

Similarly we can proceed in the second part of the proof. First, from Proposition 3.2.5 we know that $\mathcal{O}_{\widetilde{w}} = \{c \in K \mid [c]_T \in \mathcal{O}_w\}$. Now take $c \in \mathcal{O}_{\widetilde{w}}^\times$. Then $c, c^{-1} \in \mathcal{O}_{\widetilde{w}}$. This holds if and only if $[c]_T, [c^{-1}]_T \in \mathcal{O}_w$. The latter is equivalent to $[c]_T \in \mathcal{O}_w^\times$. We have shown that $\mathcal{O}_{\widetilde{w}}^\times = \{c \in K \mid [c]_T \in \mathcal{O}_w^\times\}$. Now, from Lemma 3.1.8 we obtain

$$\mathcal{M}_{\widetilde{w}} = \mathcal{O}_{\widetilde{w}} \setminus \mathcal{O}_{\widetilde{w}}^\times = \{c \in K \mid [c]_T \in \mathcal{O}_w \setminus \mathcal{O}_w^\times\} = \{c \in K \mid [c]_T \in \mathcal{M}_w\}.$$

□

Let K be a valued field with a nontrivial valuation v and let $T \subseteq \mathcal{O}_v^\times$ be a subgroup of K^*. Then the factor hyperfield K_T also admits a nontrivial valuation. Indeed, we can construct on K_T the valuation v_T. Since v is nontrivial, there exists $c \in K \setminus \mathcal{O}_v$. From Proposition 3.2.3 we have that $[c]_T \notin \mathcal{O}_T$, so $K_T \neq \mathcal{O}_T$. This shows that v_T is nontrivial. However, we have seen so far only two examples of non-factor valued hyperfields, namely the Massouros hyperfield from Proposition 1.2.12 and the hyperfield A_1 from Example 2.2.9. Both of them admit only trivial valuation. Unfortunately, for now we do not know if there exists a non-factor hyperfield which admits a nontrivial valuation.

3.3 Prüfer hyperrings

The notion of a Prüfer ring is well-known in the classical ring theory (for reference see e.g. [15] or §11 of [14] in view of valuation theory). In this section we define an analogous notion in the theory of hyperrings.

Let F be a hyperfield and R a subhyperring of F. Denote by:

- $\mathcal{S}(R)$ the set of all valuation hyperrings in F which contain the subhyperring R,

- $\mathcal{F}(R)$ the set of all valuation hyperrings $\mathcal{O}$ in F such that $\mathcal{O} = \frac{R}{S}$, where $S \subseteq R^*$ is a multiplicatively closed subset with 1. In other words, $\mathcal{O}$ is a hyperring of fractions of R.

Take $\mathcal{O} \in \mathcal{F}(R)$. Then there is a multiplicatively closed subset $S \subseteq R^*$ with 1 such that $\mathcal{O} = \frac{R}{S}$. Moreover, by Lemma 1.3.30 (a) we have that $R \simeq \frac{R}{1} \subseteq \frac{R}{S} = \mathcal{O}$. Hence $\mathcal{F}(R) \subseteq \mathcal{S}(R)$. What is more, $\mathcal{F}(R) \neq \emptyset$ if and only if F is a hyperfield of fractions of R. Indeed, if F is a hyperfield of fractions of R, then $F \in \mathcal{F}(R)$. To show the converse, let $\mathcal{O} = \frac{R}{S}$ for some multiplicatively closed subset $S \subseteq R^*$ with 1. Then for every $a \in F$ we have $a \in \frac{R}{S}$ or $a^{-1} \in \frac{R}{S}$. In both cases $a = \frac{r}{s}$, where $r, s \in R, s \neq 0$. Hence $F \subseteq \frac{R}{R^*}$. Obviously $\frac{R}{R^*} \subseteq F$, so F is a hyperfield of fractions of R.

Lemma 3.3.1. *Let R, R' be local subhyperrings in F with maximal hyperideals $\mathcal{M}_R, \mathcal{M}_{R'}$ respectively, such that $R \subseteq R'$. The following conditions are equivalent:*

(a) $\mathcal{M}_R \subseteq \mathcal{M}_{R'}$,

(b) $\mathcal{M}_R = \mathcal{M}_{R'} \cap R$.

Proof. First we will show that (a) implies (b). We assume that $\mathcal{M}_R \subseteq \mathcal{M}_{R'}$. Then $\mathcal{M}_R = \mathcal{M}_R \cap R \subseteq \mathcal{M}_{R'} \cap R$. We wish to show that $\mathcal{M}_{R'} \cap R$ is a proper hyperideal in R. Indeed, take $a, b \in \mathcal{M}_{R'} \cap R$. Then $a -_R b = (a -_F b) \cap R \subseteq R$ and $a -_R b = (a -_F b) \cap R \subseteq (a -_F b) \cap R' = a -_{R'} b \subseteq \mathcal{M}_{R'}$. Hence $a -_R b \subseteq \mathcal{M}_{R'} \cap R$. Take now $r \in R$ and $a \in \mathcal{M}_{R'} \cap R$. Then $ra \in R$ and since $r \in R \subseteq R'$, also $ra \in \mathcal{M}_{R'}$. Finally, $1 \notin \mathcal{M}_{R'} \cap R$, because $1 \notin \mathcal{M}_{R'}$. Since $\mathcal{M}_{R'} \cap R$ is a proper hyperideal of R and it contains the maximal hyperideal $\mathcal{M}_R$, we have that $\mathcal{M}_R = \mathcal{M}_{R'} \cap R$.
The reverse implication follows from the fact that $\mathcal{M}_R = \mathcal{M}_{R'} \cap R \subseteq \mathcal{M}_{R'}$. □

Definition 3.3.2. Let R, R' be local subhyperrings in F. We say that R' *dominates* R if $R \subseteq R'$ and $\mathcal{M}_R, \mathcal{M}_S$ satisfy the equivalent conditions from Lemma 3.3.1.

Lemma 3.3.3. *Let F be a hyperfield, R a subhyperring in F and $\mathcal{O}$ a valuation hyperring in F with maximal hyperideal $\mathcal{M}$ such that $\mathcal{O} \in \mathcal{S}(R)$. Then the set $\mathfrak{a} := \mathcal{M} \cap R$ is a prime hyperideal in R.*

Proof. Take $a, b \in \mathfrak{a}$. Then $a -_R b = (a -_F b) \cap R \subseteq R$ and $a -_R b = (a -_F b) \cap R \subseteq (a -_F b) \cap \mathcal{O} = a -_{\mathcal{O}} b \subseteq \mathcal{M}$, hence $a -_R b \subseteq \mathfrak{a}$. Moreover, take $r \in R$. Then $ra \in R$ and since $r \in R \subseteq \mathcal{O}$, also $ra \in \mathcal{M}$. Take now $a, b \in R$ such that $ab \in \mathfrak{a}$. Since $\mathcal{M}$ is a prime hyperideal (see Corollary 1.3.25), we have that $a \in \mathcal{M}$ or $b \in \mathcal{M}$. Hence $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$. □

Proposition 3.3.4. *Let F be a hyperfield, R a subhyperring in F and $\mathcal{O}$ a valuation hyperring in F such that $\mathcal{O} \in \mathcal{S}(R)$. Let $\mathfrak{a} = \mathcal{M}_{\mathcal{O}} \cap R$, where $\mathcal{M}_{\mathcal{O}}$ is a maximal hyperideal in $\mathcal{O}$. Then $\mathcal{O}$ dominates $R_{\mathfrak{a}}$, where $R_{\mathfrak{a}}$ is the local hyperring constructed as in Example 1.3.31 (b). Moreover, the following conditions are equivalent:*

(a) $\mathcal{O} = R_{\mathfrak{a}}$,

(b) $R_{\mathfrak{a}}$ *is a valuation hyperring in* F,

(c) $\mathcal{O} \in \mathcal{F}(R)$.

Proof. First of all we will show that $\mathcal{O}$ dominates $R_{\mathfrak{a}}$. Take elements $a, b \in R$ such that $b \notin \mathfrak{a} = \mathcal{M}_{\mathcal{O}} \cap R$. Then $\frac{a}{b} \in R_{\mathfrak{a}}$. Since $b \in R \subseteq \mathcal{O}$, so $b \notin \mathcal{M}_{\mathcal{O}}$. Hence $b \in \mathcal{O}^{\times}$ and $a \in \mathcal{O}$, so $\frac{a}{b} \in \mathcal{O}$. Thus, $R_{\mathfrak{a}} \subseteq \mathcal{O}$. Moreover,

$$\mathcal{M}_{R_{\mathfrak{a}}} = \mathfrak{a} \cdot R_{\mathfrak{a}} \subseteq \mathfrak{a} \cdot \mathcal{O} = (\mathcal{M}_{\mathcal{O}} \cap R) \cdot \mathcal{O} \subseteq \mathcal{M}_{\mathcal{O}} \cdot \mathcal{O} = \mathcal{M}_{\mathcal{O}},$$

where in the first and in the last equality we used part (b) from Example 1.3.31. Hence the condition (a) from Lemma 3.3.1 is fulfilled, so $\mathcal{O}$ dominates $R_{\mathfrak{a}}$.

Implications $(a) \Rightarrow (b)$ and $(a) \Rightarrow (c)$ are clear. To show that (b) implies (a), we use the fact that if $\mathcal{O}$ and $R_{\mathfrak{a}}$ are valuation hyperrings in F and $R_{\mathfrak{a}} \subseteq \mathcal{O}$, then from Lemma 3.1.12 we obtain that $\mathcal{M}_{\mathcal{O}} \subseteq \mathcal{M}_{R_{\mathfrak{a}}}$. Having that $\mathcal{O}$ dominates $R_{\mathfrak{a}}$, we obtain that $\mathcal{M}_{\mathcal{O}} = \mathcal{M}_{R_{\mathfrak{a}}}$, so $\mathcal{O} = R_{\mathfrak{a}}$.

To show that (c) implies (a), observe that since $\mathcal{O} \in \mathcal{F}(R)$, there is a multiplicatively closed subset $S \subseteq R^*$ with 1 such that $\mathcal{O} = \frac{R}{S}$. Observe that $S \cap \mathcal{M}_{\mathcal{O}} = \emptyset$. Indeed, assume that $a \in S \cap \mathcal{M}_{\mathcal{O}}$. Then $\frac{1}{a} \in \frac{R}{S} = \mathcal{O}$ and $a \cdot \frac{1}{a} = 1 \in \mathcal{M}_{\mathcal{O}}$, which is a contradiction. Hence $S \subseteq R \setminus \mathcal{M}_{\mathcal{O}} = R \setminus (\mathcal{M}_{\mathcal{O}} \cap R) = R \setminus \mathfrak{a}$, so $\mathcal{O} = \frac{R}{S} \subseteq R_{\mathfrak{a}}$. Since $\mathcal{O}$ dominates $R_{\mathfrak{a}}$, we obtain that $\mathcal{O} = R_{\mathfrak{a}}$. □

Proposition 3.3.5. *Let* F *be a hyperfield and* R *a subhyperring. The following are equivalent:*

(a) $R_{\mathcal{M}}$ *is a valuation hyperring for every maximal hyperideal* $\mathcal{M}$ *in* R,

(b) $R_{\mathfrak{p}}$ *is a valuation hyperring for every prime hyperideal* $\mathfrak{p}$ *in* R.

Moreover, if conditions (a) and (b) are satisfied, then also $\mathcal{S}(R) = \mathcal{F}(R)$.

Proof. First we will show that (a) implies (b). Let $\mathfrak{p}$ be a prime hyperideal in R. Since every prime hyperideal is contained in some maximal hyperideal (by Lemma 1.3.16), there exists a maximal hyperideal $\mathcal{M}$ in R such that $\mathfrak{p} \subseteq \mathcal{M}$. Hence $R \setminus \mathcal{M} \subseteq R \setminus \mathfrak{p}$, so $R_{\mathcal{M}} \subseteq R_{\mathfrak{p}}$. We assumed $R_{\mathcal{M}}$ to be a valuation hyperring, so from Lemma 3.1.13, $R_{\mathfrak{p}}$ is also a valuation hyperring.

The reverse implication follows from the fact that every maximal hyperideal is a prime hyperideal, which was shown in Corollary 1.3.25.

Now we assume that R is a subhyperring in F, which has the property that $R_{\mathfrak{p}}$ is a valuation hyperring for every prime hyperideal $\mathfrak{p}$ in R. We need to show that $\mathcal{S}(R) \subseteq \mathcal{F}(R)$. Take any $\mathcal{O} \in \mathcal{S}(R)$, i.e., $R \subseteq \mathcal{O}$ and consider the set $\mathfrak{a} = \mathcal{M}_{\mathcal{O}} \cap R$. From Lemma 3.3.3 we know that $\mathfrak{a}$ is a prime hyperideal in R. Hence, from the assumption, $R_{\mathfrak{a}}$ is a valuation hyperring. From Proposition 3.3.4 we obtain that $\mathcal{O} \in \mathcal{F}(R)$. □

Definition 3.3.6. Let F be a hyperfield. A subhyperring $R \subseteq F$ which satisfies the equivalent conditions (a) and (b) from Proposition 3.3.5 is called a *Prüfer hyperring*.

Corollary 3.3.7. *Let F be a hyperfield and $R \subseteq F$ a Prüfer hyperring. Then R is a strict subhyperring of F.*

Proof. Let $R \subseteq F$ be a Prüfer hyperring of a hyperfield F. Since R is a subhyperring of a hyperfield, it is an integral hyperdomain. From Proposition 1.3.32 we have that $R = \bigcap_{\mathcal{M} \in \mathbf{M}_R} R_{\mathcal{M}}$, where $\mathbf{M}_R$ is the set of all maximal hyperideals in R. From Proposition 3.3.5 (a) we know that each $R_{\mathcal{M}}$ is a valuation hyperring, which is a strict subhyperring in F by Proposition 3.1.7. From Lemma 1.3.3 we know that the intersection of strict subhyperrings is a strict subhyperring. Hence, R is a strict subhyperring in F. □

Chapter 4

Real valuations, real places and real holomorphy hyperrings

4.1 Compatibility between positive cones and valuations in real hyperfields

In this section we are going to introduce the notion of a valuation compatible with a positive cone in real hyperfields and prove a Baer-Krull Theorem for real hyperfields. A number of results in this section were originally published in [26].

Let K be an ordered field with a positive cone P. Since $\operatorname{char} K = 0$, it contains the rationals. Consider the set

$$A(P) := \{a \in K \mid n \pm a \in P \text{ for some } n \in \mathbb{N}\}.$$

It is a valuation ring of K and the set

$$I(P) := \{a \in K \mid \frac{1}{n} \pm a \in P \text{ for all } n \in \mathbb{N}\}$$

is its maximal ideal. The valuation associated with the valuation ring $A(P)$ is the natural valuation associated with P described in Subsection 2.2.

It is natural to ask if it is possible to construct similar sets in real hyperfields. However, the real hyperfield does not have to contain the rationals. The simplest example is the sign hyperfield from Example 2.2.4, where $1 + 1 = \{1\}$.

Let F be a hyperfield. For $n \in \mathbb{N}$, define

$$I_n := \underbrace{1 + \ldots + 1}_{n \text{ times}}.$$

Lemma 4.1.1 (Lemma 5.1, [26]). *With the notations introduced above we have, for all $n, m \in \mathbb{N}$*

$$I_n + I_m = I_{n+m} \quad \text{and} \quad I_n \cdot I_m \subseteq I_{nm}.$$

Proof. The first equality is a consequence of the axiom (C1) in Definition 1.1.7. The second inclusion follows from Remark 1.2.2. □

Take a hyperfield F with a positive cone P. Define

$$A(P) := \{a \in F \mid (I_n \pm a) \cap P \neq \emptyset \text{ for some } n \in \mathbb{N}\}$$
$$I(P) := \{a \in F \mid 1 \pm I_n \cdot a \subseteq P \text{ for all } n \in \mathbb{N}\}.$$

Lemma 4.1.2. *Let F be a real hyperfield with positive cone P.*

(a) If $a \in A(P)$ (or $a \in I(P)$), then also $-a \in A(P)$ (or $-a \in I(P)$ respectively),

(b) $I(P) \subseteq A(P)$,

(c) $1 + I(P) \subseteq P$,

(d) $0 \in A(P)$ and $0 \in I(P)$,

(e) $1 \in A(P)$.

Proof. (a) This follows from the symmetry in the definitions of $A(P)$ and of $I(P)$.

(b) Take $a \in I(P)$. In particular, $1 \pm a \subseteq P$. Hence $(I_1 \pm a) \cap P \neq \emptyset$, so $a \in A(P)$.

(c) Take $a \in I(P)$. For $n = 1$ we have $1 \pm a \subseteq P$, so $1 + I(P) \subseteq P$.

(d) Since $1 \pm I_n \cdot 0 = \{1\} \subseteq P$ for every $n \in \mathbb{N}$, we obtain that $0 \in I(P)$. From part (b) we have that also $0 \in A(P)$.

(e) First observe that $I_n + 1 = I_{n+1} \subseteq P$, so $(I_n + 1) \cap P \neq \emptyset$ for every $n \in \mathbb{N}$. Moreover, $1 \in 1 + (1 - 1) = (1 + 1) - 1 = I_2 - 1$, so $(I_2 \pm 1) \cap P \neq \emptyset$. Hence $1 \in A(P)$.

□

Proposition 4.1.3 (Proposition 5.3, [26]). *Let F be a real hyperfield with positive cone P. The set $A(P)$ is a valuation hyperring in F with the unique maximal hyperideal $I(P)$. Moreover, $I_n \subseteq A(P)$ for every $n \in \mathbb{N}$.*

Proof. First we will show that $A(P)$ is a strict subhyperring in F. The fact that $0 \in A(P)$ follows from Lemma 4.1.2 (d). Take $a, b \in A(P)$. Then there are $m, n \in \mathbb{N}$ such that $(I_m \pm a) \cap P \neq \emptyset$ and $(I_n \pm b) \cap P \neq \emptyset$. Take $c \in a + b$. If $c = 0$, then $c \in A(P)$ by what we have previously observed.

Assume that $c \in P$. Let $s \in (I_m - a) \cap P$ and $t \in (I_n - b) \cap P$. From axiom (C4) in Definition 1.1.7 we obtain that $a \in I_m - s$ and $b \in I_n - t$. Then

$$c \in a + b \subseteq I_m - s + I_n - t = I_{m+n} - (s + t),$$

where the last equality follows from Lemma 4.1.1. Therefore, there exists $u \in s + t \subseteq P$ such that $c \in I_{m+n} - u$. Hence $u \in (I_{m+n} - c) \cap P$. Since we assumed $c \in P$, we have also $(I_{m+n} + c) \subseteq P$. So $c \in A(P)$.

Assume now that $c \in -P$. Then $-c \in P$. Proceeding like in the previous case, we obtain that $-c \in A(P)$. By Lemma 4.1.2 (a) we have that $c \in A(P)$. Hence, we have proved that $a + b \subseteq A(P)$.

Now we are going to show that $ab \in A(P)$. By Lemma 4.1.2 (a) we can assume that $a, b \in P$. Let $s \in (I_m - a) \cap P$ and $t \in (I_n - b) \cap P$. Choose $p \in I_m$ and $q \in I_n$ such that $s \in p - a$ and $t \in q - b$. Using axiom (C4) in Definition 1.1.7 we obtain that $p \in a + s$ and $q \in b + t$. Using Remark 1.2.2 we obtain

$$pq \in (a + s)(b + t) \subseteq ab + (at + sb + st).$$

Observe that $at + sb + st \subseteq P$. Pick $x \in at + sb + st$ such that $pq \in ab + x$. We have

$$x \in pq - ab \subseteq I_m \cdot I_n - ab \subseteq I_{mn} - ab.$$

Hence $(I_{mn} - ab) \cap P \neq \emptyset$. Since we assumed $ab \in P$, we have also $(I_{mn} + ab) \subseteq P$. We conclude that $ab \in A(P)$. We have proven that $A(P)$ is a strict subhyperring in F.

Now let us show that $A(P)$ is a valuation hyperring. Take $a \in F \setminus A(P)$. Without loss of generlity we can assume that $a \in P$ (by Lemma 4.1.2 (a)). Then also $a^{-1} \in P$, so $1 + a^{-1} \subseteq P$. Since $1 + a \subseteq P$, but $a \notin A(P)$, we obtain that $(1 - a) \cap P = \emptyset$. Hence $a - 1 \subseteq P$ or $a = 1$. The latter is impossible, because by Lemma 4.1.2 (e), $1 \in A(P)$. Therefore, $1 - a^{-1} = a^{-1}(a - 1) \subseteq P$. We obtain that $(1 \pm a^{-1}) \cap P \neq \emptyset$, hence $a^{-1} \in A(P)$.

The fact that $I_n \subseteq A(P)$ for every $n \in \mathbb{N}$ follows from Lemma 4.1.2 (e) and the fact that $A(P)$ is a strict subhyperring in F.

It is left to show that $I(P)$ is the unique maximal hyperideal in $A(P)$. We are going to prove that $I(P) = A(P) \setminus A(P)^\times$. Take $c \in A(P) \setminus A(P)^\times$. If $c = 0$, then $0 \in I(P)$ by Lemma 4.1.2 (d). Hence, take $c \neq 0$ and assume that $c \in P$ (we can do it because of Lemma 4.1.2 (a)). Then $1 + I_n \cdot c \subseteq P$ for every $n \in \mathbb{N}$. Since also $c^{-1} \in P$, we have also that $c^{-1} + I_n \subseteq P$ for every $n \in \mathbb{N}$. Because $c \notin A(P)^\times$, we have that $c^{-1} \notin A(P)$, so $(I_n - c^{-1}) \cap P = \emptyset$ for every $n \in \mathbb{N}$. Hence $c^{-1} - I_n \subseteq P \cup \{0\}$. If $c^{-1} \in I_n$ for some $n \in \mathbb{N}$, then since $I_n \subseteq A(P)$ we obtain a contradition with the fact that $c^{-1} \notin A(P)$. Hence $c^{-1} - I_n \subseteq P$ for every $n \in \mathbb{N}$, which is equivalent to $1 - I_n \cdot c \subseteq P$ for every $n \in \mathbb{N}$. We have shown that $A(P) \setminus A(P)^\times \subseteq I(P)$. Take now $c \in I(P)$. If $c = 0$, then $0 \in A(P) \setminus A(P)^\times$ by Lemma 4.1.2 (d). Hence, take $c \neq 0$ and again assume that $c \in P$. By Lemma 4.1.2 (b) we know that $c \in A(P)$. Suppose that $c \in A(P)^\times$. Then $(I_n - c^{-1}) \cap P \neq \emptyset$ for some $n \in \mathbb{N}$.

Since $c^{-1} \in P$, we obtain $(I_n \cdot c - 1) \cap P \neq \emptyset$ for some $n \in \mathbb{N}$, which implies that $1 - I_n \cdot c \notin P$ for some $n \in \mathbb{N}$. Hence $c \notin I(P)$. This contradiction proves that $c \in A(P) \setminus A(P)^\times$. We conclude that $I(P) = A(P) \setminus A(P)^\times$. By Lemma 3.1.8 the set $I(P)$ is the unique maximal hyperideal in $A(P)$. $\square$

Definition 4.1.4 (Definition 5.5, [26]). A positive cone P of a hyperfield F is called *archimedean* if $A(P) = F$.

Example 4.1.5 (Example 5.4, [26]). Consider the real factor hyperfield $F = \mathbb{Q}_{(\mathbb{Q}^*)^2}$ introduced in Example 2.2.5. Since every positive rational is a sum of at most four nonzero squares, we obtain that $P_{(\mathbb{Q}^*)^2} = I_4$. By Proposition 4.1.3 we know that $I_4 \subseteq A(P_{(\mathbb{Q}^*)^2})$, so $P_{(\mathbb{Q}^*)^2} \subseteq A(P_{(\mathbb{Q}^*)^2})$. By Lemma 4.1.2 (a) and (d), also $-P_{(\mathbb{Q}^*)^2} \subseteq A(P_{(\mathbb{Q}^*)^2})$ and $0 \in A(P_{(\mathbb{Q}^*)^2})$. Hence $A(P_{(\mathbb{Q}^*)^2}) = F$ and $P_{(\mathbb{Q}^*)^2}$ is archimedean. Moreover, by Corollary 1.3.15, we obtain that $I(P_{(\mathbb{Q}^*)^2}) = \{[0]_{(\mathbb{Q}^*)^2}\}$.

Example 4.1.6 (Example 5.6, [26]). Consider the hyperfield of pairs $F = R_{E^+(R)}$ introduced in Example 2.2.6. It is a real hyperfield with its unique positive cone $P_{E^+(R)} = \{(s, \gamma) \mid s = 1, \gamma \in \Gamma\}$. Observe that $(1,0) + (1,0) = \{(1,0)\}$, so $I_n = \{(1,0)\}$ for every $n \in \mathbb{N}$. Hence

$$A(P_{E^+(R)}) = \{a \in F \mid ((1,0) \pm a) \cap P_{E^+(R)} \neq \emptyset\}.$$

Take $(1, \gamma) \in F^*$. Then $(1,0) + (1,\gamma) \subseteq P_{E^+(R)}$. If $\gamma > 0$, then $(1,0) - (1,\gamma) = \{(1,0)\} \subseteq P_{E^+(R)}$. If $\gamma = 0$, then $(1,0) - (1,0) = (\{(s,\gamma) \mid s \in \{-1,1\}, \gamma \geq 0\} \cup \{0\}) \cap P_{E^+(R)} \neq \emptyset$. If $\gamma < 0$, then $(1,0) - (1,\gamma) = \{(-1,\gamma)\} \subseteq -P_{E^+(R)}$. Having also Lemma 4.1.2 (a) and (d), we obtain that

$$A(P_{E^+(R)}) = \{(s,\gamma) \in F^* \mid s \in \{-1,1\}, \gamma \geq 0\} \cup \{0\} \subsetneq F.$$

Hence $P_{E^+(R)}$ is not archimedean. Similarly, using Lemma 4.1.2 (a) and (d), we can compute that

$$I(P_{E^+(R)}) = \{(s,\gamma) \in F^* \mid s \in \{-1,1\}, \gamma > 0\} \cup \{0\}.$$

Observe that the valuation associated with the valuation hyperring $A(P_{E^+(R)})$ is precisely the valuation $v_{E^+(R)}$ described in Example 3.2.4.

Proposition 4.1.7 (Proposition 5.7, [26]). *Let F be a hyperfield with valuation v and positive cone P. The following conditions are equivalent:*

(a) $A(P) \subseteq \mathcal{O}_v$,

(b) $\overline{P} := \{a + \mathcal{M}_v \mid a \in P \cap {\mathcal{O}_v}^\times\}$ *is a positive cone in* $\overline{F_v} := \mathcal{O}_v/\mathcal{M}_v$,

(c) $1 + \mathcal{M}_v \subseteq P$,

(d) if $(b+a) \cap P \neq \emptyset$ *and* $(b-a) \cap P \neq \emptyset$, *then* $v(a) \geq v(b)$.

Proof. Assume that $A(P) \subseteq \mathcal{O}_v$. To prove that (a) implies (b) we will show first that if $b \in P \cap {\mathcal{O}_v}^\times$ and $a + \mathcal{M}_v = b + \mathcal{M}_v$, then $a \in P \cap {\mathcal{O}_v}^\times$. Take $a \in \mathcal{O}_v$ and $b \in P \cap \mathcal{O}_v^\times$ such that $a + \mathcal{M}_v = b + \mathcal{M}_v$. If $a \in \mathcal{M}_v$, then $\{a\} = (a - 0) \cap \mathcal{M}_v$. From Lemma 1.3.19 we obtain that $a + \mathcal{M}_v = 0 + \mathcal{M}_v = \mathcal{M}_v$. So $b + \mathcal{M}_v = \mathcal{M}_v$. Hence $b \in \mathcal{M}_v$, which is a contradiction. Thus $a \in {\mathcal{O}_v}^\times$. We multiply the equality $a + \mathcal{M}_v = b + \mathcal{M}_v$ by $b^{-1} + \mathcal{M}_v$ (we can do that, because $b \in \mathcal{O}_v^\times$, so $b + \mathcal{M}_v \neq 0 + \mathcal{M}_v = 0_{\overline{F}_v}$). We obtain that $ab^{-1} + \mathcal{M}_v = 1 + \mathcal{M}_v$. Moreover, from Lemma 3.1.12 we have that $\mathcal{M}_v \subseteq I(P)$, because $A(P) \subseteq \mathcal{O}_v$. Thus, from Lemma 1.1.6 we obtain that $1 + \mathcal{M}_v \subseteq 1 + I(P)$. Hence, using Lemma 4.1.2 (c) we have

$$ab^{-1} + \mathcal{M}_v = 1 + \mathcal{M}_v \subseteq 1 + I(P) \subseteq P.$$

In particular $ab^{-1} \in P$ and hence $a = ab^{-1} \cdot b \in P$.

Now we will show that $\overline{P}$ is a positive cone in $\mathcal{O}_v/\mathcal{M}_v$. Take $a, b \in P \cap \mathcal{O}_v^\times$ and $a + \mathcal{M}_v, b + \mathcal{M}_v \in \overline{P}$. Then $a + \mathcal{M}_v + b + \mathcal{M}_v = \{c + \mathcal{M}_v \mid c \in a + b\}$. Since P is additively closed, $c \in P$ for every $c \in a + b$. If there is an element $c_0 \in a + b$ such that $c_0 \in \mathcal{M}_v$, then $a \in -b + c_0 \subseteq -b + \mathcal{M}_v$. Since the cosets are equivalence classes of an equivalence relation (see Proposition 1.3.20), we obtain that $-b + \mathcal{M}_v = a + \mathcal{M}_v$. Since we assumed that $a \in P$, also $-b \in P$ by the observation we made in the beggining of the proof. This is a contradiction, since $b \in P$ and $P \cap -P = \emptyset$. Hence, $c_0 \in \mathcal{O}_v^\times$, which means that $c \in \mathcal{O}_v^\times \cap P$ for every $c \in a + b$. Hence, $a + \mathcal{M}_v + b + \mathcal{M}_v \subseteq \overline{P}$. Since P is multiplicatively closed and ${\mathcal{O}_v}^\times$ is a multplicative group, we obtain that $ab \in P \cap {\mathcal{O}_v}^\times$, so $(a + \mathcal{M}_v) \cdot (b + \mathcal{M}_v) = ab + \mathcal{M}_v \in \overline{P}$. Suppose that there is $a + \mathcal{M}_v \in \overline{P} \cap -\overline{P}$. Then $a \in P \cap -P$ by the observation we made in the beggining of the proof. This is a contradiction, so $\overline{P} \cap -\overline{P} = \emptyset$. Finally, take $a + \mathcal{M}_v \in (\overline{F_v})^*$. Then $a \notin \mathcal{M}_v$, so $a \in {\mathcal{O}_v}^\times$. Consequently, $a + \mathcal{M}_v \in \overline{P}$ if $a \in P$, and $a + \mathcal{M}_v \in -\overline{P}$ if $a \in -P$. Hence $\overline{P} \cup -\overline{P} = (\overline{F_v})^*$.

Let us show now that (b) implies (c). Assume that the set $\overline{P}$ is a positive cone in $\overline{F_v}$. Since $1 \in P \cap {\mathcal{O}_v}^\times$ we have that $1 + \mathcal{M}_v \in \overline{P}$. Take $x \in 1 + \mathcal{M}_v$. Then $x + \mathcal{M}_v = 1 + \mathcal{M}_v$. By what we have proven in the first part, we obtain that $x \in P \cap {\mathcal{O}_v}^\times \subseteq P$.

To prove that (d) follows from (c), assume that $1 + \mathcal{M}_v \subseteq P$. Take $a, b \in F$ such that $(b \pm a) \cap P \neq \emptyset$. Suppose that $v(a) < v(b)$. Then $a \neq 0$ and thus $v(ba^{-1}) > 0$ by Lemma 3.1.2 (c). So $ba^{-1} \in \mathcal{M}_v$. By our assumption, $1 \pm ba^{-1} \subseteq P$. Now we consider two cases. If $a \in P$, then, multiplying the last inclusion by a, we have $a \pm b \subseteq P$. This implies that $(b - a) \cap P = \emptyset$, which is a contradiction. If $a \in -P$, then $-a \pm b \subseteq P$. Hence $(a + b) \cap P = \emptyset$, so again we obtain a contradiction.

Finally, we are going to prove the implication from (d) to (a). Take $a \in A(P)$. If $a = 0$, then $a \in \mathcal{O}_v$. Suppose that $a \in P$. Then $(I_n - a) \cap P \neq \emptyset$ for some $n \in \mathbb{N}$. Therefore, there exists $x \in I_n$ such that $(x - a) \cap P \neq \emptyset$. Moreover, $x + a \subseteq P$, because $x \in P$. Hence $v(a) \geq v(x)$. Observe that from the fact that $v(1) = 0$ (see Lemma 3.1.2 (a)) and the axiom (V3) from Definition 3.1.1 we obtain that $v(x) \geq 0$. Thus $a \in \mathcal{O}_v$. Using Lemma 3.1.2 (b) we have also that $-a \in \mathcal{O}_v$. This finishes the proof. □

Definition 4.1.8 (Definition 5.8, [26]). Let F be a hyperfield.

(a) A valuation v satisfying the equivalent conditions of Proposition 4.1.7 is called a *valuation compatible with the positive cone* P. We denote by $\mathcal{X}(F, v)$ the set of all positive cones of F compatible with v.

(b) A positive cone $\overline{P}$ is called the *positive cone induced by* P *on the residue hyperfield* $\overline{F_v} = \mathcal{O}_v/\mathfrak{m}_v$.

(c) The valuation associated with the valuation hyperring $A(P)$ is called the *natural valuation associated with* P.

(d) A valuation v is called a *real valuation* if its residue hyperfield $\overline{F_v} = \mathcal{O}_v/\mathfrak{m}_v$ is real.

Remark 4.1.9. We will use two different notations for the elements in the residue hyperfield $\overline{F_v} = \mathcal{O}_v/\mathcal{M}_v$, depending on which one will be more useful in a particular situation. Namely, we will denote the elements of $\overline{F_v}$ by $a + \mathcal{M}_v$ or by $\bar{a}$ for $a \in \mathcal{O}_v$.

Example 4.1.10. In Example 4.1.6 it was noted that the valuation $v_{E^+(R)}$ on a hyperfield $F = R_{E^+(R)}$ (described in Example 3.2.4) is associated with the valuation hyperring $A(P_{E^+(R)})$. Hence the valuation $v_{E^+(R)}$ is the natural valuation associated with the positive cone $P_{E^+(R)}$.

Lemma 4.1.11. *Every finite real hyperfield admits only archimedean positive cones.*

Proof. From Lemma 3.1.3 we know that a finite hyperfield admits only a trivial valuation. In particular, a natural valuation associated to any positive cone in F is also trivial. □

Example 4.1.12. (a) Consider the sign hyperfield $\mathbb{S} = \{-1, 0, 1\}$ with its unique positive cone $P_{\mathbb{S}} = \{1\}$. Since $\mathbb{S}$ is finite, the positive cone P is archimedean.

(b) Consider the hyperfield A_1 from Example 2.2.9 with its positive cone $P = \{1, a, a^2\}$. Since A_1 is finite, the positive cone P is archimedean.

At the end of Section 3.2, we mentioned that, at this point, we do not know whether there is a non-factor hyperfield that admits a non-trivial valuation. In particular, we do not know whether there exists a non-factor, real hyperfield with a non-archimedean positive cone (which admits a non-trivial natural valuation). A positive answer to this question would also give an answer to the open question posed at the end of Section 2.2, namely, that would be an example of an infinite, real hyperfield which is not a factor.

Lemma 4.1.13. *Let F be a hyperfield, P a positive cone in F and v a valuation compatible with P. If $a, b \in P$, then $v(c) = \min\{v(a), v(b)\}$ for every $c \in a + b$.*

Proof. Without loss of generality, assume that $v(a) \leq v(b)$. From axiom (V3) from Definition 3.1.1, we have that $v(c) \geq \min\{v(a), v(b)\} = v(a)$ for every $c \in a + b$. By the reversibility axiom, $b \in c - a$, so $(c - a) \cap P \neq \emptyset$. A positive cone is additively closed, so $c \in P$ for every $c \in a + b$. Hence $c + a \subseteq P$, so in particular $(c + a) \cap P \neq \emptyset$. By Proposition 4.1.7 (d) we have that $v(c) \leq v(a)$. Hence $v(c) = v(a)$. □

Corollary 4.1.14. *Let F be a hyperfield, P a positive cone in F and v a valuation compatible with P. If $a_1, ..., a_n \in P$, then $v(c) = \min\{v(a_1), ..., v(a_n)\}$ for every $c \in a_1 + ... + a_n$.*

Proof. We proceed by induction on n. By Lemma 4.1.13 we know that the statement is true for $n = 2$. Let us assume that the statement is true for $n = k-1$. Consider $c \in a_1 + \ldots + a_{k-1} + a_k$. This means that $c \in b + a_k$, where $b \in a_1 + \ldots + a_{k-1}$. From Lemma 4.1.13 we have that $v(c) = \min\{v(b), v(a_k)\}$. Moreover, from the induction hypothesis we have that $v(b) = \min\{v(a_1), \ldots, v(a_{k-1})\}$. Hence we obtain that $v(c) = \min\{v(a_1), \ldots, v(a_n)\}$. □

Remark 4.1.15. If $a, b \in \sum(F^*)^2$, then $v(c) = \min\{v(a), v(b)\}$ for every $c \in a + b$ and every real valuation v. This follows from Lemma 4.1.13 and Example 2.1.7.

For the moment, we turn our attention to real factor hyperfields. Let (K, P) be a real field and let $T \subseteq P$ be a multiplicative subgroup of K^*. Consider the factor hyperfield (K_T, P_T), where $P_T = \{[c]_T \mid c \in P\}$ (see Lemma 2.2.1 (b)) and let us denote:

- $A(P)$ the valuation ring in K of the natural valuation v which is associated with P,
- $A(P_T)$ the valuation hyperring in K_T of the natural valuation w which is associated with P_T,
- $\widetilde{A(P_T)} = \{a \in K \mid [a] \in A(P_T)\}$ is the valuation ring of the valuation $\widetilde{w}$ as described in Theorem 3.2.2.

In the next proposition we investigate the relation between a valuation $\tilde{w}$ which is lifted up to a field (K, P) from the natural valuation w in the factor hyperfield (K_T, P_T) (see Corollary 3.2.7) and the natural valuation in (K, P).

Proposition 4.1.16. *Let (K, P) be a real field and T a multiplicative subgroup of K^* such that $T \subseteq P$. Let $v : K \to \Gamma_1 \cup \{\infty\}$ be the natural valuation on K associated with P. Let $w : K_T \to \Gamma_2 \cup \{\infty\}$ be the natural valuation on K_T associated with P_T. Then:*

(a) $A(P) \subseteq \widetilde{A(P_T)}$,

(b) $A(P) = \widetilde{A(P_T)}$ *if and only if* $T \subseteq A(P)^\times$.

Proof. (a) Take $a \in A(P)$. This means that there is $n \in \mathbb{N}$ such that $n \pm a \in P$, so $[n+a]_T, [n-a]_T \in P_T$. Since $[n]_T \in I_n$, we have $[n \pm a]_T \in [n]_T \pm [a]_T \subseteq I_n \pm [a]_T$. Hence

$$(I_n \pm [a]_T) \cap P_T \neq \emptyset,$$

which means that $[a]_T \in A(P_T)$ and $a \in \widetilde{A(P_T)}$.

(b) Assume that $T \subseteq A(P)^\times$. We need to show that $\widetilde{A(P_T)} \subseteq A(P)$. By Lemma 3.1.12 we can equivalently show that $I(P) \subseteq \widetilde{I(P_T)}$. Take $a \in I(P)$. For every $n \in \mathbb{N}$ we consider the following set:

$$[1]_T \pm I_n[a]_T = \{[u_1 \pm u_2 a]_T \mid u_1 \in T,\ u_2 \in \underbrace{T + ... + T}_{n \text{ times}}\}.$$

Observe that $A(P)^\times$ is additively closed by Corollary 4.1.14, so also $A(P)^\times \cap P$ is additively closed. Hence we have

$$\underbrace{T + ... + T}_{n \text{ times}} \subseteq A(P)^\times \cap P$$

for every $n \in \mathbb{N}$. Hence $u_2 \in A(P)^\times \cap P$. Then $v(u_1) = v(u_2) = 0$. Since $a \in I(P)$, we have that $v(a) > 0$ and $v(u_2 a) > 0$. This implies that $u_2 a$ and $-u_2 a$ are archimedean smaller than u_1 and $u_1 \pm u_2 a \in P$. Hence for every $n \in \mathbb{N}$, $u_1 \in T$ and $u_2 \in \underbrace{T + ... + T}_{n \text{ times}}$ we have $[u_1 \pm u_2 a]_T \in P_T$, so $[1]_T \pm I_n[a]_T \subseteq P_T$ for all n. Hence $[a]_T \in I(P_T)$. This implies that $a \in \widetilde{I(P_T)}$.

Assume now that $T \not\subseteq A(P)^\times$. Then there exists $a \in T$ such that $a \notin A(P)$ or $a \in I(P)$. In the latter case, $a^{-1} \notin A(P)$ and since T is a multiplicative subgroup in K^*, we have that $a^{-1} \in T$. Hence, we conclude that there exists $a \in T$ such that $a \notin A(P)$. Since $a \in T$, we have that $w([a]_T) = w([1]_T) = 0$, so $[a]_T \in A(P_T)$. Hence $a \in \widetilde{A(P_T)}$, but $a \notin A(P)$. In particular, $A(P) \neq \widetilde{A(P_T)}$.

□

To finish this section, we are going to recall results about a positive cone in the residue hyperfield and state and prove a Baer-Krull Theorem for real hyperfields.

Proposition 4.1.17 (Proposition 5.9, [26])**.** *Let F be a real hyperfield with positive cone P. The positive cone $\overline{P} = \{\overline{a} \in \overline{F} \mid a \in P \cap A(P)^\times\}$ induced by P on the residue hyperfield $\overline{F} = A(P)/I(P)$ is archimedean.*

Proof. Let v be the natural valuation associated with the positive cone P. Observe first that if $a \in A(P)$ and $x \in (I_n + a) \cap P$ or $x \in (I_n - a) \cap P$, then $x + 1 \subseteq A(P)^\times \cap P$. Indeed, since $x \in P$ and P is additively closed, we have that $x + 1 \subseteq P$. Take $s \in x + 1$. Since $A(P)$ is additively closed, we have that $s \in A(P)$, so $v(s) \geq 0$. Moreover, from the reversibility axiom, we have that

$x \in (s-1) \cap P$, so $(s-1) \cap P \neq \emptyset$. Since $s+1 \subseteq P$, using Proposition 4.1.7 (d) and Lemma 3.1.2 (a), we obtain that $v(s) \leq v(1) = 0$. Therefore $s \in A(P)^\times \cap P$.

Take $\overline{a} \in \overline{P}$. Then $a \in A(P)$. Thus there is $n \in \mathbb{N}$ and $x, y \in F$ such that $x \in (I_n + a) \cap P$ and $y \in (I_n - a) \cap P$. Take $s \in x+1$ and $t \in y+1$. By what we have shown, we know that $s, t \in A(P)^\times \cap P$. Moreover, $s \in x+1 \subseteq (I_n+a)+1 = I_{n+1}+a$ and $t \in y+1 \subseteq (I_n-a)+1 = I_{n+1}-a$. Hence we obtain

$$\overline{s} \in (\overline{I_{n+1}} + \overline{a}) \cap \overline{P} \text{ and } \overline{t} \in (\overline{I_{n+1}} - \overline{a}) \cap \overline{P},$$

where $\overline{I_n} := \overline{1} + \ldots \overline{1} = \{\overline{x} \mid x \in I_n\}$. This proves that $\overline{a} \in A(\overline{P})$, so $\overline{P}$ is archimedean. □

Lemma 4.1.18. *Let F be a real hyperfield with positive cone P, and v a valuation compatible with P. Then v is the natural valuation associated with P if and only if $\overline{P} = \overline{P \cap \mathcal{O}_v^\times}$ is an archimedean positive cone in the residue hyperfield $\overline{F_v}$.*

Proof. Let v be the natural valuation associated with P. Then $\mathcal{O}_v = A(P)$ and $\mathcal{M}_v = I(P)$. By Proposition 4.1.17 we have that $\overline{P} = \overline{P \cap A(P)^\times}$ is an archimedean positive cone in $\overline{F}_v = A(P)/I(P)$.

Assume now that $\overline{P} = \overline{P \cap \mathcal{O}_v^\times}$ is an archimedean positive cone in the residue hyperfield $\overline{F_v}$. Suppose that $A(P) \subsetneq \mathcal{O}_v$. Take $a \in \mathcal{O}_v \setminus A(P)$ and, without loss of generality, assume that $a \in P$ (we can do so because of Lemma 4.1.2 (a)). Since $(I_n + a) \subseteq P$ for every $n \in \mathbb{N}$, we conclude that $(I_n - a) \cap P = \emptyset$ for every $n \in \mathbb{N}$. So we have $a - I_n \subseteq P \cup \{0\}$ for every $n \in \mathbb{N}$; but $0 \in a - I_n$ is impossible since otherwise, $a \in I_n \subseteq A(P)$.

Take $n \in \mathbb{N}$ and choose any $x \in a - I_n \subseteq P$. Then there exists $t \in I_n$ such that $x \in a - t$. Hence $v_P(a) < 0$, because $a \notin A(P)$, and $v_P(t) = 0$ by Corollary 4.1.14. So $v_P(a) < v_P(t)$. By Lemma 3.1.2 (d) we obtain that $v_P(x) = v_P(a) < 0$. Hence $x \notin A(P)$. Since $1 \in \mathcal{O}_v$ and $\mathcal{O}_v$ is additively closed, we have that $a - I_n \subseteq \mathcal{O}_v$, so $x \in \mathcal{O}_v$. By Lemma 3.1.12 we have that $x \notin \mathcal{M}_v$ (otherwise $x \in \mathcal{M}_v \subsetneq I(P) \subseteq A(P)$, which is a contadiction). So $x \in \mathcal{O}_v^\times$. Hence $a - I_n \subseteq \mathcal{O}_v^\times$ for every $n \in \mathbb{N}$.

Consider the residue hyperfield $\overline{F_v} = \mathcal{O}_v/\mathcal{M}_v$. Then $\overline{a} - \overline{I_n} = \overline{a - I_n} \subseteq \overline{P}$ for every $n \in \mathbb{N}$. So $\overline{a} \notin A(\overline{P})$, which means that $\overline{P}$ is not an archimedean positive cone in $\overline{F_v}$. □

Proposition 4.1.19 (Proposition 5.12, [26])**.** *Let v be a real valuation on a hyperfield F. Take a positive cone $\mathfrak{p}$ in $\overline{F_v}$. Then there exists a positive cone P in F such that P is compatible with v and $\overline{P} = \mathfrak{p}$.*

Proof. Consider the following set:

$$T = \{a \in F \mid \overline{ax^2} \in \mathfrak{p} \text{ for some } x \in F\}.$$

We are going to show that T is a prepositive cone of F. First, it is easy to see that $T \subseteq F^*$. Indeed, if $0 \in T$, then $\overline{0} = \overline{0 \cdot x^2} \in \mathfrak{p}$ for some $x \in F$, which is impossible.

Take $a, b \in T$. This means that there exist $x, y \in F$ such that $\overline{ax^2}, \overline{by^2} \in \mathfrak{p}$. We will show that T is additively closed. Since $\overline{0} \notin \mathfrak{p}$, we have that $\overline{ax^2}, \overline{by^2} \neq \overline{0}$. Hence $ax^2, by^2 \in \mathcal{O}_v^\times$. Without loss of generality, we may assume that $xy^{-1} \in \mathcal{O}_v$ (if not, then $x^{-1}y \in \mathcal{O}_v$). Take $c \in a + b$. Then

$$cx^2 \in (a+b)x^2 = ax^2 + bx^2 = ax^2(1 + a^{-1}b).$$

Observe that $a^{-1}b = (ax^2)^{-1} \cdot by^2 \cdot (xy^{-1})^2$ and $(ax^2)^{-1}, by^2, (xy^{-1})^2 \in \mathcal{O}_v$. Hence we can consider the elements $\overline{(ax^2)^{-1}}, \overline{by^2}, \overline{(xy^{-1})^2} \in \overline{F_v}$. The first two of them are in $\mathfrak{p}$ and the last one is either in $\mathfrak{p}$ or is equal to $\overline{0}$. Thus, $\overline{a^{-1}b} \in \mathfrak{p} \cup \{\overline{0}\}$. Since $\mathfrak{p}$ is additively closed, we obtain that $\overline{1 + a^{-1}b} = \overline{1} + \overline{a^{-1}b} \subseteq \mathfrak{p}$. Therefore, since $\mathfrak{p}$ is also multiplicatively closed, we have $\overline{cx^2} \in \overline{ax^2}(\overline{1 + a^{-1}b}) \subseteq \mathfrak{p}$, which implies that $T + T \subseteq T$.

To show that T is multiplicatively closed, observe that $\overline{(ab)(xy)^2} = \overline{ax^2} \cdot \overline{by^2} \in \mathfrak{p}$, so $ab \in T$. Hence $T \cdot T \subseteq T$. Take now $c \in F^*$. Then $\overline{c^2 \cdot (c^{-1})^2} = \overline{1} \in \mathfrak{p}$, so $c^2 \in T$. Thus, $(F^*)^2 \subseteq T$. We have proven that T is a prepositive cone of F.

By Corollary 2.1.12 there exists a positive cone P such that $T \subseteq P$. We are going to show that $\overline{P} = \mathfrak{p}$. Take $\overline{a} \in \overline{P}$. Then $a \in P \cap \mathcal{O}_v^\times$. Suppose that $\overline{a} \notin \mathfrak{p}$. This implies that $-\overline{a} \in \mathfrak{p}$. Since $\overline{-a \cdot 1^2} = -\overline{a} \in \mathfrak{p}$, we have $-a \in T \subseteq P$. Hence $a \in P \cap -P$, which is a contradiction. Therefore, $\overline{P} \subseteq \mathfrak{p}$. To show the converse, take $\overline{a} \in \mathfrak{p}$. Then $a \in \mathcal{O}_v^\times$. Moreover, $\overline{a \cdot 1^2} = \overline{a} \in \mathfrak{p}$, so $a \in T \subseteq P$. Thus $a \in P \cap \mathcal{O}_v^\times$, so $\overline{a} \in \overline{P}$. Hence $\overline{P} = \mathfrak{p}$. In particular, by Proposition 4.1.7 (b), this shows that P is compatible with v. □

Corollary 4.1.20. *Let F be a hyperfield and v a valuation on F. Then v is a real valuation if and only if there exists a positive cone P in F such that v is compatible with P.*

Proof. Assume that v is a real valuation on F. Then from Proposition 4.1.19 we know that there exists a positive cone P on F such that P is compatible with v. Conversely, assume that there exists a positive cone P on F such that P is compatible with v. Then from Proposition 4.1.7 (b) we have that the residue hyperfield $\overline{F_v}$ is real, so v is a real valuation. □

Lemma 4.1.21. *Let F be a hyperfield and v a real valuation on F. Then $\sum {a_i}^2 \subseteq \mathcal{O}_v$ if and only if $a_1, ..., a_n \in \mathcal{O}_v$.*

Proof. One of the implications follows from the fact that a hyperring is multiplicatively and additively closed. To prove the second implication, we use the result from Lemma 2.1.15, i.e., $\sum {a_i}^2 = 0$ if and only if $a_1 = ... = a_n = 0$. Hence, in that case the statement is true. Assume now that $\sum {a_i}^2 \neq 0$. Without loss of generality, we can assume that $a_1, ..., a_n \neq 0$. Note that $(F^*)^2 \subseteq P$ for all $P \in \mathcal{X}(F)$. In particular, $a_1^2, ..., a_n^2 \in P_v$ for the positive cone P_v, which is compatible with v. Hence, by Corollary 4.1.14 we obtain that for every $c \in \sum {a_i}^2 \subseteq \mathcal{O}_v$ we have that $0 \leq v(c) = 2 \cdot \min\{v(a_1), ..., v(a_n)\}$, so $v(a_1), ..., v(a_n) \geq 0$. □

Definition 4.1.22 ([26])**.** Let F be a real hyperfield with positive cone P. Consider the group homomorphism

$$\operatorname{sgn}_P : F^* \to \{1, -1\}$$

defined by

$$\operatorname{sgn}_P(a) := \begin{cases} 1 & \text{if } a \in P, \\ -1 & \text{if } a \in -P. \end{cases}$$

We will call it the *signature* of the positive cone P of the hyperfield F.

The positive cone P is the kernel of sgn_P.

Recall that we denote by $\mathcal{X}(F)$ the set of all positive cones in a hyperfield F and by $\mathcal{X}(F, v)$ the set of all positive cones of F compatible with a valuation v.

Theorem 4.1.23 (Theorem 5.13, [26], Baer-Krull Theorem for real hyperfields)**.** *Let v be a valuation on a real hyperfield F with value group Γ. Then there is a bijection between the set $\mathcal{X}(F, v)$ and the set $\mathcal{X}(\overline{F}) \times \operatorname{Hom}(\Gamma, \{1, -1\})$.*

Proof. Take $\mathfrak{p} \in \mathcal{X}(\overline{F})$. From Proposition 4.1.19 we know that there exists $P_{\mathfrak{p}} \in \mathcal{X}(F, v)$ such that $\overline{P_{\mathfrak{p}}} = \mathfrak{p}$. Take any $P \in \mathcal{X}(F, v)$ which also induces $\mathfrak{p}$ and define a map

$$\begin{aligned} \chi_P : \Gamma &\to \{-1, 1\} \\ v(a) &\mapsto \operatorname{sgn}_{P_{\mathfrak{p}}}(a) \cdot \operatorname{sgn}_P(a). \end{aligned}$$

We are going to show that χ_P is well-defined. If $v(a) = v(b)$, then $v(ab^{-1}) = 0$, so $ab^{-1} \in \mathcal{O}_v^\times$. Hence $\overline{ab^{-1}} \in \overline{F}$. Since P induces $\mathfrak{p}$, we have that $ab^{-1} \in P$ if and only if $\overline{ab^{-1}} \in \mathfrak{p}$. The latter is equivalent to $ab^{-1} \in P_{\mathfrak{p}}$, so $\operatorname{sgn}_P(ab^{-1}) = \operatorname{sgn}_{\mathfrak{p}}(\overline{ab^{-1}}) = \operatorname{sgn}_{P_{\mathfrak{p}}}(ab^{-1})$. Thus,

$$\chi_P(v(ab^{-1})) = \operatorname{sgn}_P(ab^{-1}) \cdot \operatorname{sgn}_{P_{\mathfrak{p}}}(ab^{-1}) = 1.$$

Since sgn_P is a group homomorphism, we obtain

$$\operatorname{sgn}_P(a) \cdot \operatorname{sgn}_{P_{\mathfrak{p}}}(a) = \operatorname{sgn}_P(b) \cdot \operatorname{sgn}_{P_{\mathfrak{p}}}(b).$$

Hence $\chi_P(v(a)) = \chi_P(v(b))$.

Now we will show that $\chi_P \in \operatorname{Hom}(\Gamma, \{1, -1\})$. Observe that

$$\begin{aligned} \chi_P(v(a) + v(b)) = \chi_P(v(ab)) &= \operatorname{sgn}_{P_{\mathfrak{p}}}(ab) \cdot \operatorname{sgn}_P(ab) \\ &= \operatorname{sgn}_{P_{\mathfrak{p}}}(a) \cdot \operatorname{sgn}_P(a) \cdot \operatorname{sgn}_{P_{\mathfrak{p}}}(b) \cdot \operatorname{sgn}_P(b) = \chi_P(v(a)) \cdot \chi_P(v(b)), \end{aligned}$$

where we used the axiom (V2) from Definition 3.1.1 and the fact that sgn_P is a group homomorphism.

Let us define the mapping

$$\Phi : \mathcal{X}(F, v) \to \mathcal{X}(\overline{F}) \times \mathrm{Hom}(\Gamma, \{-1, 1\})$$
$$P \mapsto (\overline{P}, \chi_P).$$

We will show that this map is surjective. Take $\mathfrak{p} \in \mathcal{X}(\overline{F})$ and $h \in \mathrm{Hom}(\Gamma, \{-1, 1\})$. We are going to show that there exists $P \in \mathcal{X}(F, v)$ such that $\Phi(P) = (\mathfrak{p}, h)$. Let us define

$$P := \{x \in F^* \mid (h(v(x)) = 1 \wedge x \in P_{\mathfrak{p}}) \vee (h(v(x)) = -1 \wedge x \notin P_{\mathfrak{p}})\}.$$

We need to prove that P is a positive cone in F. Take $a, b \in P$ and consider the following cases:

- **Case 1:** $a, b \in P_{\mathfrak{p}}$
  Since $P_{\mathfrak{p}}$ is multiplicatively closed, $ab \in P_{\mathfrak{p}}$. Moreover, from the definition of P we have $h(v(a)) = h(v(b)) = 1$, so using axiom (V2) from Definition 3.1.1 and the fact that h is a group homomorphism, we obtain

  $$h(v(ab)) = h(v(a) + v(b)) = h(v(a)) \cdot h(v(b)) = 1 \cdot 1 = 1.$$

  Hence $ab \in P$.

  Without loss of generality we can assume that $v(a) \leq v(b)$. Take $c \in a + b$. Then $c \in P_{\mathfrak{p}}$, since $P_{\mathfrak{p}}$ is additively closed. From the reversibility axiom, $b \in c - a$, so $(c - a) \cap P_{\mathfrak{p}} \neq \emptyset$. What is more, $c + a \subseteq P_{\mathfrak{p}}$. Since $P_{\mathfrak{p}}$ is compatible with v, from Proposition 4.1.7 (d) we conclude that $v(c) \leq v(a)$. On the other hand, from axiom (V3) from Definition 3.1.1, we have $v(c) \geq \min\{v(a), v(b)\} = v(a)$. So $v(a) = v(c)$ and $h(v(c)) = h(v(a)) = 1$. Hence $c \in P$.

- **Case 2:** $a \in P_{\mathfrak{p}}$ and $b \notin P_{\mathfrak{p}}$
  Then $ab \notin P_{\mathfrak{p}}$. Moreover, from the definition of P we have $h(v(a)) = 1$ and $h(v(b)) = -1$, so

  $$h(v(ab)) = h(v(a) + v(b)) = h(v(a)) \cdot h(v(b)) = 1 \cdot -1 = -1.$$

  Hence $ab \in P$.

  Take $c \in a + b$. If $c \in P_{\mathfrak{p}}$, then $(a + b) \cap P_{\mathfrak{p}} \neq \emptyset$. Moreover, $a - b \subseteq P_{\mathfrak{p}}$, so from Proposition 4.1.7 (d) we have that $v(a) \leq v(b)$. If $v(a) = v(b)$, then $h(v(a)) = h(v(b))$, which is not the case. Hence $v(a) < v(b)$. Thus, $v(c) = \min\{v(a), v(b)\} = v(a)$, so $h(v(c)) = h(v(a)) = 1$, which implies that $c \in P$.

  If $c \notin P_{\mathfrak{p}}$, then $-c \in (-a - b) \cap P_{\mathfrak{p}}$, so $(-a - b) \cap P_{\mathfrak{p}} \neq \emptyset$. What is more, $a - b \subseteq P_{\mathfrak{p}}$, so $v(b) = v(-b) \leq v(a)$. From the same reason as before, we have $v(b) < v(a)$ and hence $v(c) = v(b)$. We obtain $h(v(c)) = h(v(b)) = -1$. As in the previous case, $c \in P$. This proves that $a + b \subseteq P$.

- **Case 3:** $a \notin P_{\mathfrak{p}}$ and $b \notin P_{\mathfrak{p}}$

  In that case we have that $ab \in P_{\mathfrak{p}}$. Moreover, from the definition of P we have $h(v(a)) = h(v(b)) = -1$, so using axiom (V2) from Definition 3.1.1 and the fact that h is a group homomorphism, we obtain

  $$h(v(ab)) = h(v(a) + v(b)) = h(v(a)) \cdot h(v(b)) = -1 \cdot -1 = 1.$$

  Hence $ab \in P$.

  Without loss of generality we can assume that $v(a) \leq v(b)$. Take $c \in a + b$. Then $c \notin P_{\mathfrak{p}}$. From the reversibility axiom, $-b \in -c + a$, so $(-c+a) \cap P_{\mathfrak{p}} \neq \emptyset$. What is more, $-c - a \subseteq P_{\mathfrak{p}}$. Since $P_{\mathfrak{p}}$ is compatible with v, from Proposition 4.1.7 (d) we conclude that $v(-c) \leq v(a)$. Using Lemma 3.1.2 (b) we have that $v(c) \leq v(a)$. On the other hand, from axiom (V3) from Definition 3.1.1, we have $v(c) \geq \min\{v(a), v(b)\} = v(a)$. So $v(a) = v(c)$ and $h(v(c)) = h(v(a)) = -1$. Hence $c \in P$.

- **Case 4:** This is the same as Case 2, with a and b interchanged.

Now we will show that $P \cap -P = \emptyset$ and $P \cup -P = F^*$. Observe that for $x \in F^*$,

$$\begin{aligned} -x \in P &\Leftrightarrow (h(v(x)) = 1 \wedge -x \in P_{\mathfrak{p}}) \vee (h(v(x)) = -1 \wedge -x \notin P_{\mathfrak{p}}) \\ &\Leftrightarrow (h(v(x)) = 1 \wedge x \notin P_{\mathfrak{p}}) \vee (h(v(x) = -1 \wedge x \in P_{\mathfrak{p}}) \\ &\Leftrightarrow (h(v(x)) = -1 \vee x \notin P_{\mathfrak{p}}) \wedge (h(v(x)) = 1 \vee x \in P_{\mathfrak{p}}) \Leftrightarrow x \notin P. \end{aligned}$$

Therefore, P is a positive cone in F.

Take $x \in \mathcal{O}_v^\times \cap P$. Since h is a group homomorphism, $h(v(x)) = h(0) = 1$, so $x \in P_{\mathfrak{p}}$. Hence $\overline{P} = \overline{P_{\mathfrak{p}}} = \mathfrak{p}$, so by Proposition 4.1.7 (b) we have that $P \in \mathcal{X}(F, v)$.

Let us show now that $\chi_P = h$. Take $x \in F^*$ and let $x \in P$.

- If $x \in P_{\mathfrak{p}}$, then $h(v(x)) = 1$ and $\chi_P(v(x)) = \operatorname{sgn}_{P_{\mathfrak{p}}}(x) \cdot \operatorname{sgn}_P(x) = 1 \cdot 1 = 1$.
- If $x \notin P_{\mathfrak{p}}$, then $h(v(x)) = -1$ and $\chi_P(v(x)) = \operatorname{sgn}_{P_{\mathfrak{p}}}(x) \cdot \operatorname{sgn}_P(x) = -1 \cdot 1 = -1$.

Now let $x \notin P$. Then $-x \in P$.

- If $x \in P_{\mathfrak{p}}$, then $h(v(x)) = h(v(-x)) = -1$ and $\chi_P(v(x)) = \operatorname{sgn}_{P_{\mathfrak{p}}}(x) \cdot \operatorname{sgn}_P(x) = 1 \cdot -1 = -1$.
- If $x \notin P_{\mathfrak{p}}$, then $h(v(x)) = h(v(-x)) = 1$ and $\chi_P(v(x)) = \operatorname{sgn}_{P_{\mathfrak{p}}}(x) \cdot \operatorname{sgn}_P(x) = -1 \cdot -1 = 1$.

Hence $\chi_P = h$ and thus $\Phi(P) = (\overline{P}, \chi_P) = (\mathfrak{p}, h)$. We have proved that Φ is surjective.

To show that Φ is injective, suppose that $\Phi(P) = \Phi(Q)$. Then $(\overline{P}, \chi_P) = (\overline{Q}, \chi_Q)$. This means that for every $x \in F^*$,

$$\operatorname{sgn}_{P_{\mathfrak{p}}}(x) \cdot \operatorname{sgn}_P(x) = \chi_P(x) = \chi_Q(x) = \operatorname{sgn}_{P_{\mathfrak{p}}}(x) \cdot \operatorname{sgn}_Q(x).$$

Hence $\operatorname{sgn}_P(x) = \operatorname{sgn}_Q(x)$, i.e., $P = Q$. This shows that Φ is bijective and this finishes the proof. $\square$

4.2 Order relation associated with a positive cone

Let (K, P) be an ordered field. With the positive cone P one can associate a strict, linear order relation defined as follows:

$$a < b :\Leftrightarrow b - a \in P.$$

This linear order relation is compatible with addition and multiplication by positive elements in K, i.e., if $a < b$, then for any $c \in F$ we have $a + c < b + c$ and for any $c \in P$ we have $ac < bc$. On the other hand, having defined a strict, linear order relation $<$ on a field K, compatible with the operations in K, we can define a positive cone in the following way:

$$P := \{a \in K \mid a > 0\}.$$

Hence, in the case of fields, the existence of a positive cone is equivalent to the existence of a strict, linear order relation, which is compatible with addition and multiplication by positive elements in the field.

Consider the set $P_0 := P \cup \{0\}$ on K. Observe that the following is true:

$$P_0 + P_0 \subseteq P_0, \qquad P_0 \cdot P_0 \subseteq P_0, \qquad P_0 \cap -P_0 = \{0\}, \qquad P_0 \cup -P_0 = K. \tag{4.1}$$

Remark 4.2.1. Some authors (see e.g. [28], [22]) define the set P_0 to be a positive cone in an ordered field.

Also with this set one can associate a linear order relation, compatible with addition and multiplication, but in this case non-strict. It is defined as follows:

$$a \leq b :\Leftrightarrow b - a \in P_0. \tag{4.2}$$

Similarly as in the case of a strict linear order relation $<$, having a non-strict linear order relation, one can define the set P_0 on K as:

$$P_0 := \{a \in K \mid a \geq 0\}.$$

It is clear that, having the set P_0 on K, one can easily obtain a positive cone $P := P_0 \setminus \{0\}$.

Now we turn our attention again to real hyperfields. Let F be a real hyperfield with a positive cone, which we will denote by P as in the case of fields. The relation defined as

$$a < b :\Leftrightarrow b - a \subseteq P \tag{4.3}$$

is not in general a strict linear order relation, as we will see in the next example:

Example 4.2.2. Consider the factor hyperfield $F = \mathbb{Q}_{(\mathbb{Q}^*)^2}$ from Example 2.2.5 with its unique positive cone $P_{(\mathbb{Q}^*)^2} = \{[a]_{(\mathbb{Q}^*)^2} \mid a \in \mathbb{Q}^+\}$. Observe that

$$[1]_{(\mathbb{Q}^*)^2} = [2 \cdot 1 - 1 \cdot 1]_{(\mathbb{Q}^*)^2} \in [2]_{(\mathbb{Q}^*)^2} - [1]_{(\mathbb{Q}^*)^2} \text{ and } [2]_{(\mathbb{Q}^*)^2} = [1 \cdot 4 - 2 \cdot 1]_{(\mathbb{Q}^*)^2} \in [1]_{(\mathbb{Q}^*)^2} - [2]_{(\mathbb{Q}^*)^2}.$$

Hence

$$([2]_{(\mathbb{Q}^*)^2} - [1]_{(\mathbb{Q}^*)^2}) \cap P_{(\mathbb{Q}^*)^2} \neq \emptyset \text{ and } ([1]_{(\mathbb{Q}^*)^2} - [2]_{(\mathbb{Q}^*)^2}) \cap P_{(\mathbb{Q}^*)^2} \neq \emptyset,$$

which means that elements $[1]_{(\mathbb{Q}^*)^2}$ and $[2]_{(\mathbb{Q}^*)^2}$ are incomparable with respect to the order relation $<$ induced by $P_{(\mathbb{Q}^*)^2}$.

However, the relation $<$ is a strict partial order relation. Indeed, $0 \in a - a$, but $0 \notin P$, so $a - a \nsubseteq P$, which means that $a \not< a$. Moreover, if $a < b$, then $b - a \subseteq P$. Since $P \cap -P = \emptyset$, we obtain that $a - b \nsubseteq P$, so $b \not< a$. Hence the relation $<$ is asymmetric. If $a < b$ and $b < c$ then $b - a \subseteq P$ and $c - b \subseteq P$ and since P is additively closed, we obtain that $c - a \subseteq c - b + b - a \subseteq P$, so $a < c$, which shows that $<$ is transitive. What is more, the relation $<$ is compatible with multiplication by positive elements. Indeed, assume that $a < b$ and $c \in P$. Then $b - a \subseteq P$ and since P is multiplicatively closed, we obtain that $(b - a)c = bc - ac \subseteq P$, i.e., $ac < bc$.

It is worth mentioning that $<$ does not have to *behave well* with hyperaddition in the following sense:

Example 4.2.3. Consider the sign hyperfield with its unique positive cone $P_{\mathbb{S}} = \{1\}$. Observe that $-1 < 1$, because $1 - (-1) = 1 + 1 = \{1\} \subseteq P_{\mathbb{S}}$. However, adding 1 to both sides of the equality, we obtain

$$-1 + 1 = \{-1, 0, 1\} \text{ and } 1 + 1 = \{1\}.$$

Consider now the set $P_0 := P \cup \{0\}$. This set has analogous properties as the set P_0 in an ordered field, which were described in (4.1).

As it was already mentioned in Section 2.1, some authors define the set P_0 to be a positive cone in a hyperfield (see e.g. [34] and [16]). The following relation on a hyperfield F, associated with P_0, is a non-strict, partial order relation:

$$a \leq b :\Leftrightarrow b - a \subseteq P_0. \tag{4.4}$$

Indeed, $\leq$ is reflexive, because $0 \in a - a$ for every $a \in F$, so $a \leq a$. Assume that there are $a, b \in F$ such that $a \leq b$ and $b \leq a$. This means that $b - a \subseteq P_0$ and $a - b \subseteq P_0$. Since $P_0 \cap -P_0 = \{0\}$, we obtain that $b - a = a - b = \{0\}$. This implies that $a = b$, so the relation $\leq$ is antisymmetric. Take now $a, b, c \in F$ such that $a \leq b$ and $b \leq c$. Then $b - a \subseteq P_0$ and $c - b \subseteq P_0$ and since P_0 is an additively closed set, we obtain that $c - a \subseteq c - b + b - a \subseteq P_0$, so $a \leq c$. This proves that $\leq$ is transitive.

Definition 4.2.4. Let (F, P) be a real hyperfield. We define the following relation on F:

$$\begin{aligned} a \sim b &: \Leftrightarrow a \not< b \wedge b \not< a \\ &\Leftrightarrow (b - a) \cap P_0 \neq \emptyset \wedge (a - b) \cap P_0 \neq \emptyset. \end{aligned}$$

The relation $\sim$ is called the *incomparability relation with respect to* P and if $a \sim b$, then we say that a and b are *incomparable with respect to* P.

Let F be a real hyperfield. Note that for every $a, b \in F$ either $a < b$ or $b < a$ or $a \sim b$.

Example 4.2.5. The elements $[1]_{(\mathbb{Q}^*)^2}, [2]_{(\mathbb{Q}^*)^2} \in \mathbb{Q}_{(\mathbb{Q}^*)^2}$ described in Example 4.2.2 are incomparable, i.e., $[1]_{(\mathbb{Q}^*)^2} \sim [2]_{(\mathbb{Q}^*)^2}$.

Lemma 4.2.6. *Let (K, P) be a real field and (K_T, P_T) a real factor hyperfield as described in Theorem 2.2.2. Then $[c]_T \sim [1]_T$ if and only if there are $s_1, s_2 \in T$ such that $s_1 \leq_P c \leq_P s_2$, where $\leq_P$ is the order relation associated with P_0.*

Proof. Assume that $[c]_T \sim [1]_T$. This is equivalent to the existence of $s_1, s_2 \in T$ such that $[c - s_1]_T \in ([c]_T - [1]_T) \cap P_0$ and $[s_2 - c]_T \in ([1]_T - [c]_T) \cap P_0$. The latter is equivalent to $c - s_1, s_2 - c \in P_0$, i.e., $s_1 \leq_P c \leq_P s_2$. □

Proposition 4.2.7. *Let (F, P) be a real hyperfield. An incomparability relation $\sim$ on F is an equivalence relation.*

Proof. Take $a \in F$. Then $0 \in a - a$ and $0 \in P_0 \cap -P_0$, so $a \sim a$. Hence the relation $\sim$ is reflexive. Symmetry of the relation $\sim$ is clear. To prove transitivity, take $a, b, c \in F$ and assume that $a \sim b$ and $b \sim c$. This means that there exist $x, y \in P_0$ such that $x \in b - a$ and $y \in c - b$. Using the reversibility axiom we obtain that $b \in x + a$ and hence $y \in c - x - a$. So $(x + y) \cap (c - a) \neq \emptyset$. Since $x + y \subseteq P$, we conclude that $(c - a) \cap P_0 \neq \emptyset$. On the other hand, there exist $z, t \in P_0$ such that $z \in a - b$ and $t \in b - c$. Again by the reversibility axiom, we have that $b \in a - z$ and $t \in a - z - c$. Hence $(t + z) \cap (a - c) \neq \emptyset$ and $(a - c) \cap P_0 \neq \emptyset$. This shows that $a \sim c$. □

Lemma 4.2.8. *Let (F, P) be a real hyperfield. The incomparability relation $\sim$ has the following properties:*

(a) If $b \in P$ and $a \in -P_0$, then $a < b$. In particular, $a \nsim b$.

(b) $a \sim 0$ if and only if $a = 0$.

(c) If $a \sim b$, then $-a \sim -b$.

(d) If $a \sim b$ then $ca \sim cb$ for every $c \in F$.

(e) If $a \sim b$ and $a < c$, then $b < c$. If $a \sim b$ and $d < a$, then $d < b$,

Proof. (a) If $a \in -P$, then $-a \in P$ and since P is additively closed, we obtain that $b - a \subseteq P$. If $a = 0$, then $b - 0 = \{b\} \subseteq P$. In both cases, $a < b$. Since $(a - b) \cap P_0 = \emptyset$, we have $a \nsim b$.

(b) From reflexivity of the relation $\sim$ we know that $a \sim a$ for every $a \in F$. In particular, $0 \sim 0$. Conversely, assume that $a \sim 0$. Then $\{a\} = a - 0 \subseteq P_0$ and $\{-a\} = 0 - a \subseteq P_0$. Since $P_0 \cap -P_0 = \{0\}$, we have that $a = 0$.

(c) Take $a, b \in F$ such that $a \sim b$. Since $(a-b) \cap P_0 \neq 0$, $(-b-(-a)) \cap P_0 \neq 0$. Similarly, since $(b-a) \cap P_0 \neq 0$, $(-a-(-b)) \cap P_0 \neq 0$. Hence $-a \sim -b$.

(d) Assume that $a \sim b$. If a or b is equal to 0, then from part (b) we know that $a = b = 0$ and the statement is clearly true. Assume now that both a and b are nonzero. We know that $(a-b) \cap P_0 \neq \emptyset$ and $(b-a) \cap P_0 \neq \emptyset$. If $c = 0$, then $cb - ca = 0 - 0 = \{0\} = 0 - 0 = ca - cb$, so $(ca-cb) \cap P_0 \neq \emptyset$ and $(cb-ca) \cap P_0 \neq \emptyset$. If $c \in P$, then

$$\emptyset \neq c(a-b) \cap cP_0 \subseteq (ca-cb) \cap P_0,$$
$$\emptyset \neq c(b-a) \cap cP_0 \subseteq (cb-ca) \cap P_0.$$

If $c \in -P$, then $-c \in P$ and

$$\emptyset \neq (-c)(a-b) \cap (-c)P_0 = (cb-ca) \cap (-c)P_0 \subseteq (cb-ca) \cap P_0,$$
$$\emptyset \neq (-c)(b-a) \cap (-c)P_0 = (ca-cb) \cap (-c)P_0 \subseteq (ca-cb) \cap P_0.$$

Hence, $ca \sim cb$.

(e) Assume that $a \sim b$ and $a < c$. Suppose that $b \not< c$. Then either $b \sim c$ or $c < b$. If $b \sim c$, then from transitivity of the incomparability relation, we have that $a \sim c$. This is a contradiction. Hence, $c < b$. But from transitivity of $<$ we conclude that $a < b$, which also is a contradiction. So $b < c$. The proof of the second part of the statement is analogous.

□

Proposition 4.2.9. *Let (F, P) be a finite real hyperfield. Then $1 \sim a$ for every $a \in P$.*

Proof. Take $a \in P$. Then by Lemma 2.1.4 there exists $x \in P$ such that $a \in 1 + x$. By the reversibility axiom we have that $x \in a - 1$, so $(a-1) \cap P_0 \neq \emptyset$. On the other hand, again by the same lemma, there exists $y \in P$ such that $a^{-1} \in 1 + y$. Multiplying by a and using the reversibility axiom, we obtain $ya \in 1 - a$. Since $ya \in P$, we have that $(1-a) \cap P_0 \neq \emptyset$, so $1 \sim a$. This finishes the proof. □

Lemma 4.2.10. *Consider the real hyperfield (F, P) and let w be the natural valuation associated with P. If two elements $a, b \in F$ are incomparable, i.e. $a \sim b$, then $w(a) = w(b)$.*

Proof. If $a = b$, then $w(a) = w(b)$. Hence, we assume that $a \neq b$. From Lemma 4.2.8 (b) we have that in that case $a, b \neq 0$ and from Lemma 4.2.8 (a) we conclude that either $a, b \in P$ or $a, b \in -P$. From Lemma 4.2.8 (c) we know that if $a \sim b$, then also $-a \sim -b$. Hence, we can assume that $a, b \in P$. From Proposition 4.1.7 (d) we have that $(b-a) \cap P \neq \emptyset$ and $(b+a) \cap P \neq \emptyset$ implies $w(a) \geq w(b)$. On the other hand, by Proposition 4.1.7 (d), $(a-b) \cap P \neq \emptyset$ and $(a+b) \cap P \neq \emptyset$ implies $w(a) \leq w(b)$. Hence $w(a) = w(b)$. □

Let (F, P) be a real hyperfield. The antichain in F are the equivalence classes of the incomparability relation on F. We will denote as A_a the antichain which contains the element $a \in F$. Hence, $b \in A_a$ if and only if $a \sim b$, which is also equivalent to $A_a = A_b$. Observe that a hyperfield is linearly ordered if and only if each antichain contains only one element.
Consider a family of antichains

$$\Upsilon_P := \{A_a \mid a \in P\}$$

on F. We introduce the operation on Υ_P as follows:

$$A_a \circ A_b := A_{ab}$$

and the following relation on Υ_P:

$$A_a \prec A_b :\Leftrightarrow a < b.$$

Proposition 4.2.11. *The operation $\circ$ and the relation $\prec$ are well-defined, and the tuple $(\Upsilon_P, \circ, A_1, \prec)$ is an ordered abelian group.*

Proof. First we will show that the operation $\circ$ is well-defined. Take $a_1, a_2, b_1, b_2 \in P$ and assume that $a_1 \sim a_2$ and $b_1 \sim b_2$. Then $a_1 b_1 \sim a_2 b_1 \sim a_2 b_2$, where we used Lemma 4.2.8 (d). Hence,

$$A_{a_1} \circ A_{b_1} = A_{a_1 b_1} = A_{a_2 b_2} = A_{a_2} \circ A_{b_2},$$

so the operation is well-defined. Now we will show that $(\Upsilon_P, \circ, A_1)$ is an abelian group. Associativity and commutativity of the operation $\circ$ follows from the same laws of multiplication in F. Take $A_a \in \Upsilon_P$. Then

$$A_a \circ A_1 = A_{a \cdot 1} = A_a = A_{1 \cdot a} = A_1 \circ A_a,$$

so A_1 is the identity element. Observe that

$$A_a \circ A_{a^{-1}} = A_{a \cdot a^{-1}} = A_1 = A_{a^{-1} \cdot a} = A_{a^{-1}} \circ A_a.$$

Since $a \in P$ implies $a^{-1} \in P$, we obtain that $A_{a^{-1}} \in \Upsilon_P$ is the inverse element for A_a. Hence Υ_P is an abelian group.

Now we will show that the relation $\prec$ is a linear order relation on Γ. First we will show that this relation is well-defined. Let $A_{a_1} = A_{a_2}$ and $A_{b_1} = A_{b_2}$. This means that $a_1 \sim a_2$ and $b_1 \sim b_2$. Assume that $a_1 < b_1$. Using Lemma 4.2.8 (e) twice, we obtain that $a_2 < b_2$, which means that $\prec$ is well-defined. The relation $\prec$ is irreflexive, antisymmetric and transitive, because the same is true for $<$. It is also compatible with the operation $\circ$, since $<$ is compatible with multiplication by positive elements. It is left to show that $\prec$ is linear. Take $A_a, A_b \in \Upsilon_P$. Assume that $A_a \not\prec A_b$ and $A_b \not\prec A_a$. Then $a \not< b$ and $b \not< a$, which means that $a \sim b$. So $A_a = A_b$. This finishes the proof. □

Remark 4.2.12. Let F be a real hyperfield and take $a \in P$. Then $A_a \subseteq P$. Indeed, take $b \in A_a$. Then $a \sim b$. If $b \in -P_0$, then we obtain a contradiction to Lemma 4.2.8 (a). Thus, $b \in P$. Hence, the family of antichains Υ_P is a group of antichains with positive elements.

In fact, we can introduce a linear order relation on the family of all antichains on a hyperfield F in the same way as it was done on Υ_P:

$$A_a \prec A_b :\Leftrightarrow a < b.$$

Hence, a real hyperfield is the union over a chain of antichains. Since we have proved that Υ_P is an ordered abelian group, we conclude that the length of the chain of antichains in F is either equal to 3 or it is infinite. This is because the ordered group is either trivial or infinite. In the case of the length equal to 3, we obtain a chain of the form $A_{-1} \prec A_0 \prec A_1$. Moreover, $A_1 = P$ and $A_{-1} = -P$. We will call this type of a chain of antichains *a short chain of antichains*. Note that a real hyperfield is linearly ordered if and only if every antichain has only one element.

Example 4.2.13. Consider the sign hyperfield $\mathbb{S} = \{-1, 0, 1\}$. It admits a unique positive cone $P_{\mathbb{S}} = \{1\}$. This hyperfield is linearly ordered with $-1 < 0 < 1$. Hence, the chain of antichains is a short one and each antichain contains only one element.

Example 4.2.14. Every finite real hyperfield is the union of a short chain of antichains. This follows from Proposition 4.2.9.

Example 4.2.15 ([20])**.** Consider the factor hyperfield $F = \mathbb{Q}_{(\mathbb{Q}^*)^2}$ from Example 2.2.5 where $P_{(\mathbb{Q}^*)^2} = \{[a]_{(\mathbb{Q}^*)^2} \mid a \in \mathbb{Q}^+\}$ is its unique positive cone. In this hyperfield any two positive elements are incomparable, as well as any two negative elements, so the chain of antichains is a short one.

Example 4.2.16. Consider the hyperfield of pairs $F := R_{E^+(R)}$ from Example 2.2.6. It admits a unique positive cone $P_{E^+(R)} = \{(s, \gamma) \mid s = 1, \gamma \in \Gamma\}$. This hyperfield is linearly ordered, so every antichain contains a single element. The chain of antichains is the following:

$$... \prec A_{(-1,-\gamma)} \prec ... \prec A_{(-1,0)} \prec ... \prec A_{(-1,\gamma)} \prec ... \prec A_0 \prec ... \prec A_{(1,\gamma)} \prec ... \prec A_{(1,0)} \prec ... \prec A_{(1,-\gamma)} \prec ...$$

where $\gamma > 0$.

The sign hyperfield and the hyperfield of pairs are stringent hyperfields. The following proposition gives the general result about the order relation in such hyperfields.

Proposition 4.2.17 (Proposition 10, [20], Proposition 3.20, [36])**.** *If F is a real stringent hyperfield, then F is linearly ordered.*

Proof. Take two distinct elements $a, b \in F$. Then $a - b = \{c\}$ for some $c \in F^*$. If $c \in P$, then $b < a$. If $c \in -P$, then $a < b$. Hence F is linearly ordered. □

Example 4.2.18. Let $(K, <)$ be an ordered field. Consider the field $K((X))$ of formal power series of one variable. Every nonzero element $f \in K((X))$ can be written in the following form

$$f = \sum_{i=n}^{\infty} b_i X^i, b_n \neq 0.$$

The field $K((X))$ is real and the set

$$P = \{f \in K((X)) \mid b_n > 0\}$$

is a positive cone. The natural valuation associated with P is the following:

$$v : K((X)) \to \mathbb{Z} \cup \{\infty\}$$

$$0 \mapsto \infty,$$

$$f \mapsto n.$$

For $k \in \mathbb{N}_{>1}$ consider the subgroup

$$T_k = \{f \in K((X)) \mid n \in k\mathbb{Z}, b_n > 0\}$$

and the factor hyperfield $F_k := K((X))_{T_k}$. Since $T_k \subseteq P$, we know from Theorem 2.2.2 that F_k is real and $P_{T_k} = \{[f]_{T_k} \mid f \in P\}$ is a positive cone in F_k. We are going to show that F_k is the union of a short chain of antichains and each nonzero antichain has k elements. Take $f, g \in P$. Note that $[f]_{T_k} = [g]_{T_k}$ if and only if $v(\frac{f}{g}) \in k\mathbb{Z}$, i.e., $v(f) \equiv v(g) \pmod{k}$. We will show that if $v(f) \not\equiv v(g) \pmod{k}$, then $[f]_{T_k} \sim [g]_{T_k}$. Assume that $v(f) \not\equiv v(g) \pmod{k}$ and $v(f) < v(g)$. Since v is compatible with P, we have that $f - g \in P$, so $[f - g]_{T_k} \in ([f]_{T_k} - [g]_{T_k}) \cap P_{T_k}$. Take $l \in \mathbb{Z}$ such that $v(f) + k \cdot l > v(g)$. Since v is surjective, we can find $t_0 \in T_k$ with $v(t_0) = k \cdot l$. Hence $v(f \cdot t_0) = v(f) + k \cdot l > v(g)$, so $g - ft_0 \in P$. This means that $[g - ft_0]_{T_k} \in ([g]_{T_k} - [f]_{T_k}) \cap P_{T_k}$. This proves that $[f]_{T_k} \sim [g]_{T_k}$, so all the elements from P_{T_k} are in the same antichain. Similarly we can prove the same result for $f, g \in -P$. This finishes the proof that this factor hyperfield is the union of a short chain of antichains. Moreover, since for $f, g \in P$ we have that $[f]_{T_k} \neq [g]_{T_k}$ if and only if $v(f) \not\equiv v(g) \pmod{k}$, we obtain that each nonzero antichain contains exactly k elements.

Example 4.2.19. Consider the field of iterated formal power series in two variables and real coefficients $\mathbb{R}((X))((Y))$. Every nonzero element $f \in \mathbb{R}((X))((Y))$ can be written in the following form

$$f = \sum_{i=n_f}^{\infty} (\sum_{j=m_i}^{\infty} b_{ij} X^j) Y^i, \quad b_{n_f m_{n_f}} \neq 0.$$

The field $\mathbb{R}((X))((Y))$ is real with positive cone

$$P = \{f \in \mathbb{R}((X))((Y)) \mid b_{n_f m_{n_f}} > 0\}.$$

The valuation

$$v : \mathbb{R}((X))((Y)) \to \mathbb{Z} \times \mathbb{Z} \cup \{\infty\},$$

$$0 \mapsto \infty,$$

$$f \mapsto (n_f, m_{n_f}),$$

where $\mathbb{Z} \times \mathbb{Z}$ is ordered lexicographically, is called the canonical valuation of the power series field $\mathbb{R}((X))((Y))$ (see Section 4.2.3 in [25]).

We wish to show that v coincides with the natural valuation associated with P, i.e., that $\mathcal{O}_v = A(P)$. Let us compute the valuation ring $A(P) = \{f \in \mathbb{R}((X))((Y)) \mid \exists_{p \in \mathbb{N}} \ p \pm f \in P\}$. Take $f \in \mathbb{R}((X))((Y))$ and assume that $f \in P$. Then $p + f \in P$ for every $p \in \mathbb{N}$. Now we set $g = p - f$ and consider several cases. If $n_f < 0$ or $n_f = 0$ and $m_{n_f} < 0$, then $b_{n_g m_{n_g}} = -b_{n_f m_{n_f}} < 0$, so $p - f \in -P$. If $n_f > 0$ or $n_f = 0$ and $m_{n_f} > 0$, then $b_{n_g m_{n_g}} = p > 0$, so $p - f \in P$. If $n_f = 0$ and $m_{n_f} = 0$, then $b_{n_f m_{n_f}} \in \mathbb{R}$ and there exists $p \in \mathbb{R}$ such that $p > b_{n_f m_{n_f}}$. Then $b_{n_g m_{n_g}} = p - b_{n_f m_{n_f}} > 0$, so $p - f \in P$. We obtain that $A(P) = \{f \in \mathbb{R}((X))((Y)) \mid (n_f, m_{n_f}) \geq (0, 0)\} = \mathcal{O}_v$. This shows that v is the natural valuation associated with P.

For $k \in \mathbb{N}_{>1}$ consider the factor hyperfield $F_k := \mathbb{R}((X))((Y))_{T_k}$, where

$$T_k = \{f \in \mathbb{R}((X))((Y)) \mid n_f = 0, m_0 \in k\mathbb{Z}, b_{0m_0} > 0\}.$$

Then $v({T_k}^*) = \{0\} \times k\mathbb{Z}$. Since $T_k \subseteq P$, we know from Theorem 2.2.2 that F_k is real and $P_{T_k} = \{[f]_{T_k} \mid f \in P\}$ is a positive cone in F_k. We are going to show that this hyperfield is the union of a chain of antichains in which each nonzero antichain has k elements and the group $\Upsilon_{P_{T_k}}$ of antichains with positive elements is isomorphic to $\mathbb{Z}$.

Take $f, g \in P$. Observe that $[f]_{T_k} = [g]_{T_k}$ if and only if $v(\frac{f}{g}) = (0, kz), z \in \mathbb{Z}$. Let $v(f) = (n_f, m_{n_f})$ and $v(g) = (n_g, m_{n_g})$. Then $(n_f, m_{n_f}) = (0, kz) + (n_g, m_{n_g})$. Hence, $[f]_{T_k} = [g]_{T_k}$ if and only if $n_f = n_g$ and $m_{n_f} \equiv m_{n_g} (\mathrm{mod}\ k)$.

Assume that $n_f < n_g$. Recall that $[f]_{T_k} - [g]_{T_k} = \{[f - gt]_{T_k} \mid t \in T_k\}$. Take any $t \in T_k$ and consider the element $f - gt \in \mathbb{R}((X))((Y))$. Observe that

$$\begin{aligned} f - gt &= \sum_{i=n_f}^{\infty} (\sum_{j=m_i}^{\infty} a_{ij} X^j) Y^i - \sum_{i=n_g}^{\infty} (\sum_{j=m_i}^{\infty} b_{ij} X^j) Y^i \cdot \sum_{i=0}^{\infty} (\sum_{j=(m_t)_i}^{\infty} c_{ij} X^j) Y^i \\ &= \sum_{i=n_f}^{\infty} (\sum_{j=m_i}^{\infty} a_{ij} X^j) Y^i - \sum_{i=n_g}^{\infty} (\sum_{j=m_i+(m_t)_i}^{\infty} d_{ij} X^j) Y^i = \sum_{i=n_f}^{\infty} (\sum_{j=m_i}^{\infty} e_{ij} X^j) Y^i, \end{aligned}$$

where

$$d_{ij} = \sum_{l=n_g}^{i} \sum_{k=m_l}^{j} b_{lk} c_{i-l, j-k}.$$

$$e_{ij} = \begin{cases} a_{ij} & \text{if } i < n_g \text{ or } i = n_g \text{ and } j < m_{n_g} + (m_t)_0 \\ a_{ij} - d_{ij} & \text{if } i \geq n_g \text{ or } i = n_g \text{ and } j \geq m_{n_g} + (m_t)_0. \end{cases}$$

Note that $e_{n_f m_{n_f}} = a_{n_f m_{n_f}} > 0$, so $f - gt \in P$. Hence $[f]_{T_k} - [g]_{T_k} = \{[f - gt]_{T_k} \mid t \in T_k\} \subseteq P_{T_k}$, so $[f]_{T_k} > [g]_{T_k}$.

If $n_f = n_g$ and $m_{n_f} \not\equiv m_{n_g} (\text{mod } k)$ then $[f]_{T_k} \sim [g]_{T_k}$. Indeed, assume that $n := n_f = n_g$, $m_{n_f} \not\equiv m_{n_g} (\text{mod } k)$ and $m_{n_f} < m_{n_g}$. Then $v(f) < v(g)$. Since v is compatible with P, this means that $f - g \in P$. So $[f-g]_{T_k} \in P_{T_k}$ and $[f-g]_{T_k} \in [f]_{T_k} - [g]_{T_k}$. Hence $([f]_{T_k} - [g]_{T_k}) \cap P_{T_k} \neq \emptyset$. On the other hand, take $l \in \mathbb{Z}$ such that $m_{n_f} + k \cdot l > m_{n_g}$. Since v is surjective, there exists $t_0 \in T_k$ with $v(t_0) = (0, k \cdot l)$. Hence $v(g) = (n, m_{n_g}) < (n, m_{n_f}) + (0, k \cdot l) = v(ft_0)$. This implies that $[g - ft_0]_{T_k} \in P_{T_k}$. Also $[g - ft_0]_{T_k} \in [g]_{T_k} - [f]_{T_k}$, so $([g]_{T_k} - [f]_{T_k}) \cap P_{T_k} \neq \emptyset$. Hence $[f]_{T_k} \sim [g]_{T_k}$.

Take $f \in \mathbb{R}((X))((Y))$. Denote by $\rho_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ the projection on the first coordinate, i.e., $\rho_1(v(f)) = n_f$ and consider the following map:

$$\begin{aligned} \mu : \Upsilon_{P_{T_k}} &\to \mathbb{Z} \\ A_{[f]_{T_k}} &\mapsto \rho_1(v(f)) = n_f. \end{aligned}$$

We are going to show that μ is a group isomorphism. Take $[g]_{T_k} \in P_{T_k}$ such that $A_{[f]_{T_k}} = A_{[g]_{T_k}}$. This holds if and only if $[f]_{T_k} \sim [g]_{T_k}$. The latter implies $n_f = n_g$. Hence, μ is well-defined and injective. We have that $\mu(a)$ is the projection onto the first coordinate of $v(a)$. Hence, since the valuation v on the field $\mathbb{R}((X))((Y))$ is surjective, also μ is surjective. Finally, we compute

$$\mu(A_{[f]_{T_k}} \circ A_{[g]_{T_k}}) = A_{[f]_{T_k} \cdot [g]_{T_k}} = n_{f \cdot g} = n_f + n_g = \mu([f]_{T_k}) + \mu([g]_{T_k}),$$

where the third equality follows from axiom (V2) for v. Hence, μ is a group isomorphism, so $\Upsilon_{P_{T_k}} \simeq \mathbb{Z}$. Moreover, since if $m_{n_f} \not\equiv m_{n_g} \pmod{k}$, then $[g]_{T_k} \neq [f]_{T_k}$, there are exactly k elements in any antichain with positive elements.

Similarly, we can show the same results for antichains with negative elements.

A Hahn series is a type of a formal power series, introduced by Hahn in [17]. Let K be a field and G an ordered abelian group. The field $K[[t^G]]$ of Hahn series in the indeterminate t is the set of formal expressions of the form

$$f = \sum_{i \in G} c_i t^i,$$

where $c_i \in K$ and the support $\text{supp } f := \{i \in G \mid c_i \neq 0\}$ is well-ordered. The addition works as follows:

$$\sum_{i \in G} c_i t^i + \sum_{i \in G} d_i t^i = \sum_{i \in G} (c_i + d_i) t^i$$

and the multiplication:

$$\sum_{i \in G} c_i t^i \cdot \sum_{i \in G} d_i t^i = \sum_{i \in G} \sum_{j+k=i} c_j d_k t^i.$$

Take $0 \neq f \in K[[t^G]]$ and denote $\mathrm{lc}(f) := c_i$ and $\mathrm{lt}(f) := c_i t^i$, where $i := \min \mathrm{supp}\, f$. The field $K[[t^G]]$ is a valued field (see Chapter 4 in [25]) with the valuation defined as follows:

$$\begin{aligned} v : K[[t^G]] &\to G \cup \{\infty\} \\ 0 &\mapsto \infty, \\ f &\mapsto \min \mathrm{supp}\, f. \end{aligned}$$

Moreover, if K is an ordered field with a positive cone P, then $K[[t^G]]$ is also an ordered field and it admits the positive cone $P_t := \{f \in K[[t^G]] \mid \mathrm{lc}(f) \in P\}$. Indeed, take $f, g \in P_t$. If $\min \mathrm{supp}\, f \neq \min \mathrm{supp}\, g$, then without loss of generality we can assume that $\min \mathrm{supp}\, f < \min \mathrm{supp}\, g$. In that situation, $\mathrm{lc}(f+g) = \mathrm{lc}(f) \in P$. If $\min \mathrm{supp}\, f = \min \mathrm{supp}\, g$, then $\mathrm{lc}(f+g) = \mathrm{lc}(f) + \mathrm{lc}(g) \in P$, because $P + P \subseteq P$. In both cases we obtain that $f + g \in P_t$. Moreover, $\mathrm{lc}(f \cdot g) = \mathrm{lc}(f) \cdot \mathrm{lc}(g) \in P$, so $f \cdot g \in P_t$, since $P \cdot P \subseteq P$. The axioms $P_t \cap -P_t = \emptyset$ and $P_t \cup -P_t = K[[t^G]]$ follow from the same axioms for P. Hence P_t is a positive cone in $K[[t^G]]$.

Example 4.2.20. Consider the Hahn series field in one variable $K[[Y^G]]$, where G is an ordered abelian group (written additively) and K is an ordered field. Then for every element

$$f = \sum_{i \in G} c_i Y^i \in K[[Y^G]]$$

the support $\mathrm{supp}\, f = \{i \in G \mid c_i \neq 0\}$ is well-ordered. Let $K = \mathbb{R}((X))$. We obtain the field $\mathbb{R}((X))[[Y^G]]$. Every nonzero element $f \in \mathbb{R}((X))[[Y^G]]$ can be written in the following form

$$f = \sum_{i \in G} (\sum_{j=m_i}^{\infty} b_{ij} X^j) Y^i,$$

where $b_{nm_n} \neq 0$ for $n = \min \mathrm{supp}\, f$. The field $\mathbb{R}((X))[[Y^G]]$ is real with positive cone

$$P = \{f \in \mathbb{R}((X))[[Y^G]] \mid b_{nm_n} > 0\}.$$

The canonical valuation of $\mathbb{R}((X))[[Y^G]]$ is

$$\begin{aligned} v : \mathbb{R}((X))[[Y^G]] &\to G \times \mathbb{Z} \cup \{\infty\}, \\ 0 &\mapsto \infty, \\ f &\mapsto (n, m_n), \end{aligned}$$

where $\mathbb{Z} \times G$ is ordered lexicographically. As in Example 4.2.19, one can prove that it is equal to the natural valuation associated with P. For $k \in \mathbb{N}_{>1}$ consider the factor hyperfield $F_k := \mathbb{R}((X))[[Y^G]]_{T_k}$, where

$$T_k = \{f \in \mathbb{R}((X))[[Y^G]] \mid n = 0, m_0 \in k\mathbb{Z}, b_{0m_0} > 0\}.$$

Since $T_k \subseteq P$, we know from Theorem 2.2.2 that F_k is real and $P_{T_k} = \{[f]_{T_k} \mid f \in P\}$ is a positive cone in F_k. Similarly as in Example 4.2.19 one can show that this hyperfield is the union of a chain of antichains in which each nonzero antichain has k elements and the group $\Upsilon_{P_{T_k}}$ of antichains with positive elements is isomorphic to G.

In the next subsection we are going to investigate how, in general, chains of antichains associated with partial order relations look in real factor hyperfields.

4.2.1 Order relation in factor hyperfields

In this subsection we focus our attention on real factor hyperfields, in particular, on order relations and chains of antichains.

Let (K_T, P_T) be a real factor hyperfield. From Lemma 4.2.10 we know that elements of K_T which belong to the same antichain have the same value with respect to the natural valuation associated to P_T. We are going to show under which conditions on T the converse is also true. Let us start with the following lemma.

Let K be a real field with positive cone P and let $\leq_P$ be the order relation associated with $P_0 = P \cup \{0\}$. Assume that $a_1, \ldots, a_n \in K$ such that $a_1 \leq_P \ldots \leq_P a_n$. Then we define

$$\max_P\{a_1, \ldots, a_n\} := a_n.$$

Lemma 4.2.21. *Let (K, P) be a real field with $T \subseteq P$ and T is a multiplicative subgroup of K^*. Let (K_T, P_T) be a real factor hyperfield. If $[c]_T \in A(P_T)^\times \cap P_T$, then there exists $t \in T$ and $n \in \mathbb{N}$ such that $c - (nt)^{-1}, nt - c \in P$.*

Proof. Take $[c]_T \in A(P_T)^\times \cap P_T$. Then, from Theorem 2.2.2, $c \in P$. From the definition of $A(P_T)^\times$, there exist $m, m' \in \mathbb{N}$ such that

$$(I_m \pm [c]_T) \cap P_T \neq \emptyset \text{ and } (I_{m'} \pm [c^{-1}]_T) \cap P_T \neq \emptyset.$$

Note that

$$I_m - [c]_T = \{[t_1 + \ldots + t_m - c]_T \mid t_i \in T, i \in \{1, \ldots, m\}\}$$
$$I_{m'} - [c]_T = \{[s_1 + \ldots + s_{m'} - c]_T \mid s_i \in T, i \in \{1, \ldots, m'\}\}.$$

Hence there exist $[t_1+\ldots+t_m-c]_T \in P_T$ and $[s_1+\ldots+s_{m'}-c]_T \in P_T$ for some $t_1, \ldots, t_n, s_1, \ldots, s_{m'} \in T$. From Theorem 2.2.2 we obtain that

$$t_1 + \ldots + t_m - c \in P \text{ and } s_1 + \ldots + s_{m'} - c^{-1} \in P.$$

Let $t := \max_P\{t_1, \ldots, t_n, s_1, \ldots, s_{m'}\}$ and $n := \max\{m, m'\}$. We obtain

$$nt - c \in P \text{ and } nt - c^{-1} \in P.$$

Observe that $c \cdot (nt)^{-1} \in P$, so $c - (nt)^{-1} = (nt - c^{-1}) \cdot (c \cdot (nt)^{-1}) \in P$. We conclude that there exist $t \in T$ and $n \in \mathbb{N}$ such that $nt - c \in P$ and $c - (nt)^{-1} \in P$. □

Proposition 4.2.22. *Consider a real factor hyperfield K_T with a positive cone P_T. Let P be the positive cone in the field K as described in Theorem 2.2.2. Then*

$$T \not\subseteq 1 + I(P) \Leftrightarrow \forall [c]_T \in A(P_T)^\times \cap P_T : [c]_T \sim [1]_T.$$

Proof. Take $[c]_T \in A(P_T)^\times \cap P_T$. From Lemma 4.2.21 there is $t \in T$ and $n \in \mathbb{N}$ such that $(nt)^{-1} <_P c <_P nt$.

Assume first that $T \not\subseteq A(P)^\times$, so in particular $T \not\subseteq 1 + I(P)$. Then there is $s \in T$ such that $v(s) < 0$, where v is the natural valuation on K associated with the positive cone P. Then s is archimedean larger than n, because $v(n) = 0$, so in particular $s >_P n$. Without loss of generality, we can assume that $s \geq_P t$. Indeed, if $v(t) \geq 0$, then s is archimedean larger than t, so $s >_P t$. If $v(t) < 0$, then we can take $s = t$. Hence, we obtain

$$s^{-2} <_P (nt)^{-1} <_P c <_P nt <_P s^2.$$

Using Lemma 4.2.6 we conclude that $[c]_T \sim [1]_T$.

Assume now that $T \subseteq A(P)^\times$ and $T \not\subseteq 1 + I(P)$. Since $T \subseteq A(P)$, we can consider the image of T in the residue field $\overline{K}$. The residue field $\overline{K}$ is archimedean ordered, which follows from the classical theory of ordered fields (see e.g. Section 1.3 in [35]). We conclude that $\overline{T} \neq \{\overline{1}\}$. Hence there is $s \in T$ such that $\overline{s} >_{\overline{P}} \overline{1}$, where $<_{\overline{P}}$ is a linear order relation on $\overline{K}$ associated with the positive cone $\overline{P} = \{\overline{c} \mid c \in P\}$. Let $u := \max_P\{t, s\}$. Then $u \in T$ and $\overline{u} >_{\overline{P}} \overline{1}$. Moreover, there exists $k \in \mathbb{N}$ such that $\overline{u^k} >_{\overline{P}} \overline{nt}$. We conclude

$$\overline{u^{-k}} <_{\overline{P}} \overline{(nt)^{-1}} \leq_{\overline{P}} \overline{c} \leq_{\overline{P}} \overline{nt} <_{\overline{P}} \overline{u^k},$$

so $u^{-k} <_P c <_P u^k$. Hence $[c]_T \sim [1]_T$.

For the converse implication, assume that $T \subseteq 1 + I(P)$. Then $[2]_T \neq [1]_T$ and $[1]_T, [2]_T \in A(P_T)^\times \cap P_T$. Recall that $[2]_T - [1]_T = \{[2 - t]_T \mid t \in T\}$. Take any $2 - t_0$ with $t_0 \in T$. Then $\overline{2 - t_0} = \overline{2} - \overline{t_0} = \overline{2} - \overline{1} = \overline{1} >_{\overline{P}} \overline{0}$. Hence $2 - t_0 \in P$, so $[2]_T - [1]_T \subseteq P_T$. So $[2]_T >_{P_T} [1]_T$, where $<_{P_T}$ is the partial order relation on K_T associated with the positive cone P_T. In particular, $[2]_T \nsim [1]_T$. □

Corollary 4.2.23. *Let K_T be a real factor hyperfield with a positive cone P_T. Let P be the positive cone in the field K as described in Theorem 2.2.2. Let w be the natural valuation associated with P_T. Then $T \not\subseteq 1 + I(P)$ if and only if for all $[a]_T, [b]_T \in P_T$ such that $(w([a]_T) = w([b]_T)$ we have $[a]_T \sim [b]_T$.*

Proof. Let $[a]_T, [b]_T \in P_T$ be such that $w([a]_T) = w([b]_T)$. Then $w([ab^{-1}]_T) = 0$, so $[ab^{-1}]_T \in A(P_T)^\times \cap P_T$. Hence

$$[a]_T \sim [b]_T \Leftrightarrow [ab^{-1}]_T \sim [1]_T \Leftrightarrow T \nsubseteq 1 + I(P),$$

where the first equivalence follows from Lemma 4.2.8 (d) and the second from Proposition 4.2.22. □

Corollary 4.2.24. *Let (K, P) be a real field and let T be a multiplicative subgroup in K^* such that $T \nsubseteq 1 + I(P)$. Consider the real factor hyperfield (K_T, P_T). Let w be a natural valuation associated with P_T. Then*

$$w(K_T^*) \simeq \Upsilon_{P_T},$$

where Υ_{P_T} is the group of antichains with positive elements, desribed in Proposition 4.2.11, associated with $<_{P_T}$.

Proof. Let $T \nsubseteq 1 + I(P)$ and consider the map

$$\begin{aligned} \mu : \Upsilon_{P_T} &\to w(K_T^*), \\ A_{[c]_T} &\mapsto w([c]_T), \end{aligned}$$

where $[c]_T \in P_T$. First we will show that μ is well-defined. Take $[c_1]_T, [c_2]_T \in P_T$ such that $A_{[c_1]_T} = A_{[c_2]_T}$. This means that $[c_1]_T \sim [c_2]_T$. From Lemma 4.2.10 we have that $w([c_1]_T) = w([c_2]_T)$. This shows that μ is well-defined. Now we will show that μ is a bijection. Take $[c_1]_T, [c_2]_T \in P_T$ such that $w([c_1]_T) = w([c_2]_T)$. From Corollary 4.2.23 we obtain that $[c_1]_T \sim [c_2]_T$, so $A_{[c_1]_T} = A_{[c_2]_T}$. Hence, μ is injective. Now we will show that μ is surjective. Take $\gamma \in wK_T$. Since w is surjective, there exists $[c]_T \in K_T$ such that $w([c]_T) = \gamma$. From Lemma 3.1.2 (b) we know that $w([c]_T) = w(-[c]_T)$, so we can assume that $[c]_T \in P_T$. Hence, $\mu(A_{[c]_T}) = \gamma$. Finally, we compute

$$\mu(A_{[c_1]_T} \circ A_{[c_2]_T}) = \mu(A_{[c_1]_T \cdot [c_2]_T}) = w([c_1]_T \cdot [c_2]_T) = w([c_1]_T) + w([c_2]_T) = \mu(A_{[c_1]_T}) + \mu(A_{[c_2]_T}).$$

This completes the proof that the groups Υ_{P_T} and $w(K_T^*)$ are isomorphic. □

Example 4.2.25. Consider the hyperfield of pairs $F := R_{E^+(R)}$ from Example 2.2.6. It admits a unique positive cone $P_{E^+(R)} = \{(1, \gamma) \mid \gamma \in \Gamma\}$, where Γ is the value group of the natural valuation in R associated with its unique positive cone $P = (R^*)^2$. The natural valuation associated with $P_{E^+(R)}$ is $v_{E^+(R)}$ (see Example 3.2.4 and Example 4.1.10). Observe that $E^+(R) \nsubseteq 1 + I(P)$, so by Corollary 4.2.24 we have that $\Upsilon_{P_{E^+(R)}} \simeq \Gamma$.

Example 4.2.26. Consider the real factor hyperfield $F_k = \mathbb{R}((X))((Y))_{T_k}$ from Example 4.2.19 with the positive cone $P_{T_k} = \{[f]_{T_k} \mid f \in P\}$, where $P = \{f \in \mathbb{R}((X))((Y)) \mid b_{n_f m_{n_f}} > 0\}$. We have seen that $\Upsilon_{P_{T_k}} \simeq \mathbb{Z}$. Observe that $T_k \nsubseteq 1 + I(P)$. Hence, by Corollary 4.2.24 we have that $\Upsilon_{P_{T_k}} \simeq w(F_k^*)$, which means that $w(F_k^*) \simeq \mathbb{Z}$, where w is the natural valuation on F_k associated

with P_{T_k}. Recall that $\rho_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is the projection on the first coordinate, i.e., $\rho_1(v(f)) = n_f$ for $f \in \mathbb{R}((X))((Y))$. We will show that the map

$$\begin{aligned} w : F_k &\to \mathbb{Z} \cup \{\infty\}, \\ [f]_{T_k} &\mapsto \rho_1(v(f)) = n_f, \\ [0]_{T_k} &\mapsto \infty \end{aligned}$$

is the natural valuation associated with P_{T_k}, so $w(F_k^*) = \mathbb{Z}$. Recall that $v(f) = (n_f, m_{n_f})$, where v is the natural valuation on $\mathbb{R}((X))((Y))$ associated with the positive cone P. First we will to show that w is well-defined. Take $g \in [f]_{T_k}$. This means that there is $t \in T_k$ such that $g = ft$. Since $n_t = 0$, we obtain that

$$v(g) = v(ft) = v(f) + v(t) = (n_f, m_{n_f}) + (0, m_{n_g}) = (n_f, m_{n_f} + m_{n_g}).$$

Hence, $w([g]_{T_k}) = w([f]_{T_k})$. We have that w is the projection onto the first coordinate of the value of the representant of the equivalence class with respect to v. Hence, since v is surjective, also w is surjective. Now we will prove that w is a valuation. Axiom (V1) from Definition 3.1.1 follows from the definition of w. Axiom (V2) follows from the same axiom for v. It is left to show that axiom (V3) holds. Take $[f]_{T_k} \in [g]_{T_k} + [h]_{T_k}$. Then $f = g + ht$ for some $t \in T$. Observe that

$$v(g + ht) \geq \min\{v(g), v(ht)\} = \min\{(n_g, m_{n_g}), (n_{h\cdot t}, m_{n_{h\cdot t}}\} = \min\{(n_g, m_{n_g}), (n_h, m_{n_h} + m_{n_t})\},$$

where we used the axioms for the valuation v and the fact that $n_t = 0$. Hence

$$w([f]_{T_k}) = w([g + ht]_{T_k}) = n_{g+ht} \geq \min\{n_g, n_h\} = \min\{w([g]_{T_k}), w([h]_{T_k})\}.$$

It is left to show that $\mathcal{O}_w = A(P_{T_k})$. Observe that $[1]_{T_k} + [1]_{T_k} = \{[1]_{T_k}\}$, so $I_n = \{[1]_{T_k}\}$ for every $n \in \mathbb{N}$. Hence $A(P_{T_k}) = \{[f]_{T_k} \in F_k \mid ([1]_{T_k} \pm [f]_{T_k}) \cap P_{T_k} \neq \emptyset\}$. Take $[f]_{T_k} \in P_{T_k}$. Then $[1]_{T_k} + [f]_{T_k} \subseteq P_{T_k}$. Now we will focus on the set $[1]_{T_k} - [f]_{T_k} = \{[t - f]_{T_k} \mid t \in T_k\}$. For $t \in T$, consider the element

$$t - f = \sum_{i=0}^{\infty}(\sum_{j=(m_t)_i}^{\infty} a_{ij}X^j)Y^i - \sum_{i=n_f}^{\infty}(\sum_{j=m_i}^{\infty} b_{ij}X^j)Y^i = \sum_{i=n'}^{\infty}(\sum_{j=m'_i}^{\infty} c_{ij}X^j)Y^i$$

where $a_{0(m_t)_0}, b_{n_f m_{n_f}} \in P$. Let us consider the following cases. If $w([f]_{T_k}) < 0$, then $n_f < 0$. In this case $n' = n_f, {m'}_{n'} = m_{n_f}$ and $c_{n'{m'}_{n'}} = c_{n_f m_{n_f}} = -b_{n_f m_{n_f}} \in -P$. Hence $t - f \in -P$ for every $t \in T$, so $([1]_{T_k} - [f]_{T_k}) \cap P_{T_k} = \emptyset$. If $w([f]_{T_k}) > 0$, then $n_f > 0$. In this case $n' = 0, {m'}_{n'} = (m_t)_0$ and $c_{n'{m'}_{n'}} = c_{0(m_t)_0} = a_{0(m_t)_0} > 0$. Hence $t - f \in P$ for every $t \in T$, so $([1]_{T_k} - [f]_{T_k}) \subseteq P_{T_k}$. If $w([f]_{T_k}) = 0$, then $n_f = 0$. Since v is surjective, there exists $t_0 \in T$ such that $m_{t_0} > m_{n_f}$. Consider the element $t_0 - f$. Then $n' = 0, {m'}_{n'} = m_{t_0}$, so $t_0 - f \in P$. Hence $([1]_{T_k} - [f]_{T_k}) \cap P_{T_k} \neq \emptyset$. We conclude that $\mathcal{O}_w = \{[f]_{T_k} \in F_k \mid w([f]_{T_k}) \geq 0\} = A(P_{T_k})$. We have proved that w is the natural valuation associated with P_{T_k} and $w(F_k^*) = \mathbb{Z}$.

Example 4.2.27. Consider the real factor hyperfield $F_k = \mathbb{R}((X))[[Y^G]]_{T_k}$ from Example 4.2.20 with the positive cone $P_{T_k} = \{[f]_{T_k} \mid f \in P\}$, where $P = \{f \in \mathbb{R}((X))[[Y^G]] \mid b_{nm_n} > 0\}$. We have seen that $\Upsilon_{P_{T_k}} \simeq G$. Observe that $T_k \nsubseteq 1 + I(P)$. Hence, by Corollary 4.2.24 we have that $\Upsilon_{P_{T_k}} \simeq w(F_k^*)$, which means that $w(F_k^*) \simeq G$, where w is the natural valuation on F_k associated with P_{T_k}. In fact one can show, proceeding similarly as in Example 4.2.26, that the following map:

$$\begin{aligned} w : F_k &\to G \cup \{\infty\} \\ [f]_{T_k} &\mapsto n_f \\ [0]_{T_k} &\mapsto \infty \end{aligned}$$

is the natural valuation associated with P_{T_k}, so $w(F_k^*) = G$.

Remark 4.2.28. Consider the field $F = \mathbb{R}((X))((Y))$ and the factor hyperfield $F_k = \mathbb{R}((X))((Y))_{T_k}$ form Example 4.2.26. Note that $T_k \nsubseteq A(P)^\times$. Hence, by Proposition 4.1.16 we obtain that $A(P) \subsetneq \widetilde{A(P)}$. Indeed, we have computed that $A(P) = \{f \mid n_f \geq 0, m_{n_f} \geq 0\}$ and $\widetilde{A(P)} = \{f \mid n_f \geq 0, m_{n_f} \in \mathbb{Z}\}$, so $\widetilde{w}$ is coarser on F than v. We can observe a similar situation in the field $\mathbb{R}((X))[[Y^G]]$ from Example 4.2.27.

Now we will see an example of a real hyperfield with positive cone P where the group Υ_P of antichains with positive elements is not isomorphic to the value group of the natural valuation associated with P.

Example 4.2.29. Let R be an ordered field with a non-archimedean positive cone P. Let v be the natural valuation associated with P. For $\gamma \in vR$ consider the following subgroup in R:

$$T_\gamma = 1 + I(P)^\gamma := \{1 + a \mid v(a) > \gamma\}.$$

Let $F_\gamma := R_{T_\gamma}$ and consider the two elements $[1]_{T_\gamma}, [2]_{T_\gamma} \in F_\gamma$. Let w be the natural valuation on F_γ associated with P_{T_γ}. Since $T_\gamma \subseteq 1 + I(P)$ we can proceed exactly in the same way as in the end of the proof of Proposition 4.2.22 to show that $[1]_{T_\gamma} \nsim [2]_{T_\gamma}$. This means that $A_{[1]_{T_\gamma}} \neq A_{[2]_{T_\gamma}}$. On the other hand, $[1]_{T_\gamma}, [2]_{T_\gamma} \in A(P_T)^\times$, so $w([1]_{T_\gamma}) = 0 = w([2]_{T_\gamma})$. Hence $\Upsilon_{P_{T_\gamma}} \ncong w(F_\gamma^*)$.

Corollary 4.2.30. *Let $F = K_T$ be a real factor hyperfield which is not a field, with an archimedean positive cone P_T. Let $<_{P_T}$ be the order relation associated with P_T. Then the chain of antichains associated with $<_{P_T}$ is a short chain.*

Proof. Let $F = K_T$ be a real factor hyperfield which is not a field, with an archimedean positive cone P_T. Let w be the natural valuation associated with P_T. Then $w(F^*) = \{0\}$. If $T \subseteq 1 + I(P)$, from Proposition 4.1.16 (b) we have that $A(P) = \widetilde{A(P_T)}$. Since $A(P_T) = F$, we have that $\widetilde{A(P_T)} = K$, so $A(P) = K$ and $I(P) = \{0\}$. Hence, $T = \{1\}$, so K_T is a field and this is a contradiction to our assumption on F. Hence $T \nsubseteq 1 + I(P)$. Then by Corollary 4.2.23 for $[a]_T, [b]_T \in P_T$ we have

$w([a]_T) = w([b]_T)$ if and only if $[a]_T \sim [b]_T$, so every two positive elements are incomparable. Hence $A_{[1]_T} = P_T$, so the chain of antichains is a short chain.

□

To finish this subsection, we will prove two main results about the structure of real factor hyperfields K_T. We will see under which condition the positive cone P_T is archimedean and what determines the number of elements in every nonzero antichain. In particular, we are going to see which property the subgroup T needs to have to ensure that the partial order relation associated with P_T is a linear order relation.

Definition 4.2.31. Let $(S, \leq)$ be an ordered set and let $T \subseteq S$. The convex hull of T in S with respect to $\leq$ is the set

$$\text{conv}(T) = \{x \in S \mid \exists_{s,t \in T} \quad s \leq x \leq t\}.$$

Remark 4.2.32. It is well known that the convex hull of T in S with respect to $\leq$ is the intersection of all convex subsets of S which contain the set T. Hence, it is the smallest convex set with respect to inclusion which contains the subset T.

Lemma 4.2.33. *Let K be a real field with positive cone P. Let $T \subseteq P$ be a multiplicative group and let* conv(T) *be the convex hull of T in K. Consider the real factor hyperfield (K_T, P_T). Then*

(a) conv$(T) = \{c \in K \mid [c]_T \sim [1]_T\}$,

(b) conv$(T) \subseteq \widetilde{A(P_T)^\times} \cap P$,

(c) if $T \not\subseteq 1 + I(P)$, then conv$(T) = \widetilde{A(P_T)^\times} \cap P$,

(d) conv(T) *is a multiplicative subgroup in P,*

(e) T is a multiplicative subgroup in conv(T).

Proof. (a) By Lemma 4.2.6 we know that $[c]_T \sim [1]_T$ if and only if there are $s, t \in T$ such that $s \leq_P c \leq_P t$. The latter is equivalent to $c \in \text{conv}(T)$.

(b) Take $c \in \text{conv}(T)$. From the definition of conv(T) and the convexity of P we have that $[c]_T \in P_T$. By part (a), we know that $[c]_T \sim [1]_T$, hence $([c]_T - [1]_T) \cap P_T \neq \emptyset$ and $([1]_T - [c]_T) \cap P_T \neq \emptyset$. Since $[1]_T + [c]_T \subseteq P_T$, from Proposition 4.1.7 (d) we have on the one hand that $w([c]_T) \geq w([1]_T) = 0$ and on the other hand that $w([c]_T) \leq w([1]_T) = 0$, where w is the natural valuation associated with P_T. Hence, $[c]_T \in A(P_T)^\times$. Therefore, $c \in \widetilde{A(P_T)^\times} \cap P$.

(c) Assume that $T \not\subseteq 1 + I(P)$. Due to part (b), it is enough to show that $\widetilde{A(P_T)^\times} \cap P \subseteq \text{conv}(T)$. Take $c \in \widetilde{A(P_T)^\times} \cap P$. Then $[c]_T \in A(P_T)^\times \cap P_T$. By Proposition 4.2.22 we have that $[c]_T \sim [1]_T$. By part (a) we obtain that $c \in \text{conv}(T)$.

(d) Take $x, y \in \operatorname{conv}(T)$. We are going to show that $xy^{-1} \in \operatorname{conv}(T)$. There exist $s_1, s_2, t_1, t_2 \in T$ such that $s_1 \leq_P x \leq_P t_1$ and $s_2 \leq_P y \leq_P t_2$. Then $t_2^{-1} \leq_P y^{-1} \leq_P s_2^{-1}$. Suppose that $xy^{-1} - t_1 s_2^{-1} \in P$. Observe that $t_1 - x, s_2^{-1} - y^{-1} \in P$, but

$$(t_1 - x)(s_2^{-1} - y^{-1}) + (xy^{-1} - t_1 s_2^{-1}) = x(y^{-1} - s_2^{-1}) + y^{-1}(x - t_1) \notin P$$

This is a contradiction. Hence $t_1 s_2^{-1} - xy^{-1} \in P \cup \{0\}$, so $xy^{-1} \leq_P t_1 s_2^{-1}$. Similarly we show that $s_1 t_2^{-1} \leq_P xy^{-1}$. Since $s_1 t_2^{-1}, t_1 s_2^{-1} \in T$, we obtain that $xy^{-1} \in \operatorname{conv}(T)$.

(e) This follows from the fact that $T \subseteq \operatorname{conv}(T)$, the set T is a multiplicative subgroup in P and $\operatorname{conv}(T)$ is a multiplicative subgroup in P by part (d). □

Proposition 4.2.34. *Let (K, P) be a real field. Consider a multiplicative subgroup $T \subseteq P$ and the real factor hyperfield $F = K_T$ with the positive cone P_T. Then P_T is an archimedean positive cone if and only if P is an archimedean positive cone in K or T is cofinal in K.*

Proof. Assume first that P_T is archimedean in K_T. This means that $A(P_T)^\times = {K_T}^*$, so $\widetilde{A(P_T)}^\times = K^*$. If $T \not\subseteq 1+I(P)$, then by Lemma 4.2.33 (c), $\operatorname{conv}(T) = P$. Hence T is cofinal in K. If $T \subseteq 1+I(P)$, then in particular $T \subseteq A(P)^\times$. Hence by Proposition 4.1.16 (b) we have $A(P) = \widetilde{A(P_T)}$. Since P_T is archimedean, $A(P_T) = K_T$, so $A(P) = \widetilde{A(P_T)} = K$, which proves that P is archimedean in K.

For the converse, assume first that T is cofinal in K. Then $\operatorname{conv}(T) = P$, so $P = \operatorname{conv}(T) \subseteq \widetilde{A(P_T)}^\times \cap P \subseteq P$, where for the first inclusion we used Lemma 4.2.33 (b). Hence $\widetilde{A(P_T)}^\times \cap P = P$. Since $\widetilde{A(P_T)}^\times$ is symmetric, we obtain $\widetilde{A(P_T)}^\times = P \cup -P$. Hence $\widetilde{A(P_T)} = K$. This means that $A(P_T) = K_T$, so P_T is archimedean. If P is archimedean, then $A(P) = K$. By Proposition 4.1.16 (a) we know that $K = A(P) \subseteq \widetilde{A(P_T)}$, so $\widetilde{A(P_T)} = K$. This implies that $A(P_T) = K_T$, so again P_T is archimedean. □

Example 4.2.35. In Example 4.2.18 the subgroup T_k is cofinal in $K((X))$. The factor hyperfield (F_k, P_{T_k}) is archimedean, although the field $(K((X)), P)$ is non-archimedean. Indeed, in (F_k, P_{T_k}) any two positive elements are incomparable. From Lemma 4.2.10 we know that if two elements are incomparable, their values with respect to the natural valuation w are equal. In particular each positive element has the same value as the element 1, hence $w(a) = w(1) = 0$ for every $a \in P$. Using Lemma 3.1.2 (b), we obtain that $w(F_k^*) = \{0\}$, so $A(P_{T_k}) = F_k$.

Proposition 4.2.36. *Let (K_T, P_T) be a real factor hyperfield. Consider the chain of antichains associated with the partial order relation $<_{P_T}$. The number of elements in the antichain $A_i, i \neq 0$ is equal to the order of the quotient group $\operatorname{conv}(T)/T$.*

Proof. From Lemma 4.2.33 (e) we know that T is a multiplicative subgroup of $\operatorname{conv}(T)$, so we can

consider the quotient group $\text{conv}(T)/T$. Consider the map

$$\begin{aligned} \mu : \text{conv}(T)/T &\to A_1 \\ cT &\mapsto [c]_T. \end{aligned}$$

We are going to show that μ is a bijection. Take $c \in \text{conv}(T)$ and consider $cT \in \text{conv}(T)/T$. First we will show that $\mu(cT) \in A_1$. From Lemma 4.2.33 (a) we know that $[c]_T \sim [1]_T$, so $[c]_T \in A_1$. Now we will show that μ is well-defined and injective. Take $c_1, c_2 \in \text{conv}(T)$ such that $c_1T = c_2T$. This holds if and only if $c_1{c_2}^{-1} \in T$. The latter is equivalent to $[c_1{c_2}^{-1}]_T = [1]_T$, so $[c_1]_T = [c_2]_T$. To see that μ is surjective, take $[c_0]_T \in A_1$. Then $\mu(c_0T) = [c_0]_T$, so μ is indeed surjective. This shows that μ is a bijection. Hence, the number of elements in the antichain A_1 is equal to the order of the quotient group $\text{conv}(T)/T$. Every nonzero antichain $A_i, i \neq 0$ has the same number of elements, so this finishes the proof. □

Corollary 4.2.37. *Let (K_T, P_T) be a real factor hyperfield. The partial order relation in K_T is a linear order relation if and only if T is a convex subgroup in K^*.*

Proof. Let K_T be a real factor hyperfield. The partial order relation in K_T is a linear order relation if and only if every antichain A_i in the chain of antichains associated with $<_{P_T}$ contains exactly one element. The latter is equivalent to $|\text{conv}(T)/T| = 1$ by Proposition 4.2.36. This holds if and only if $T = \text{conv}(T)$, i.e., T is a convex subgroup in K^*. □

4.2.2 Compatibility and convexity

In the case of real fields a valuation v is said to be compatible with a positive cone P if the valuation ring $\mathcal{O}_v$ is convex with respect to the linear order relation $<$ associated with P.

The strict partial order relation defined in (4.3) associated to a positive cone allows us to consider the notion of convexity in hyperfields. We are going to investigate the relation between valuations compatible with positive cones in hyperfields (in the sense of Proposition 4.1.7) and the convexity of valuation hyperrings. Namely, we will prove that if a valuation v on a hyperfield is compatible with a positive cone P in the sense of Proposition 4.1.7, then the valuation hyperring $\mathcal{O}_v$ is convex with respect to the order relation $<$ associated with P, but the converse is not necessary true.

Lemma 4.2.38. *Let F be a real hyperfield with positive cone P. Take $a, b \in F$ such that $a < b$. If $a \in P$, then $b \in P$ and if $b \in -P$, then $a \in -P$.*

Proof. Let $a < b$. Then $b - a \subseteq P$. Assume first that $a \in P$. Since P is additively closed, we have

$$b = b + 0 \subseteq b + (a - a) = a + (b - a) \subseteq P.$$

Now assume that $b \in -P$, i.e., $-b \in P$. Again, from the fact that P is additively closed,

$$-a = -a + 0 \subseteq -a + (b - b) = -b + (b - a) \subseteq P.$$

Hence $a \in -P$. □

Lemma 4.2.39 (Lemma 5.10, [26]). *Let v be a valuation on the real hyperfield F, compatible with a positive cone P. Then $\mathcal{O}_v$ is convex with respect to P, i.e., if $a, b \in \mathcal{O}_v$ and $a < x < b$, then $x \in \mathcal{O}_v$.*

Proof. Assume that $a, b \in \mathcal{O}_v$ and $a < x < b$. From the definition of the order relation, we have $b - x \subseteq P$ and $x - a \subseteq P$.

- If $x \in P$, then $b \in P$ by Lemma 4.2.38. Thus $b + x \subseteq P$. Since also $b - x \subseteq P$, from Proposition 4.1.7 (d), $v(x) \geq v(b) \geq 0$, so $x \in \mathcal{O}_v$.
- If $x \in -P$, then $a \in -P$ by Lemma 4.2.38. Thus $a + x \subseteq -P$, so $-a - x \subseteq P$. Since also $-a + x \subseteq P$, from Proposition 4.1.7 (d), $v(x) \geq v(-a) = v(a) \geq 0$, so $x \in \mathcal{O}_v$.

□

Definition 4.2.40. A valuation v on a field or a hyperfield is of rank 1 if its value group is archimedean.

Proposition 4.2.41 (Proposition 5.11, [26]). *Let v be a valuation on a real field K and let P be a positive cone of K. Take $T = P \cap \mathcal{O}_v^\times$ and consider the factor hyperfield K_T with the valuation v_T induced by v and the positive cone P_T induced by P (see Theorem 2.2.2 and Theorem 3.2.2). Then:*

(a) The valuation v is compatible with P if and only if v_T is compatible with P_T.

(b) If v is a rank 1 valuation which is not compatible with P, then any two elements in K_T which are positive (with respect to P_T) are incomparable. In particular, $\mathcal{O}_{v_T}$ is convex with respect to P_T, while v_T is not compatible with P_T.

Proof. (a) Assume first that P is compatible with v. Then $1 + \mathcal{M}_v \subseteq P$, which follows from Proposition 4.1.7 (c) and Example 2.1.2. Take $x \in \mathcal{M}_v$. Since $T \subseteq P$, we have that $1t_1 + xt_2 = t_1(1 + xt_1^{-1}t_2) \in P$ for every $t_1, t_2 \in T$. Hence, using Lemma 3.2.8,

$$[1]_T + [x]_T = \{[1t_1 + xt_2]_T \mid t_1, t_2 \in T\} \subseteq P_T.$$

Therefore, we obtain that P_T is compatible with v_T by Proposition 4.1.7 (c).

To prove the converse implication, assume that P_T is compatible with v_T, i.e., $[1]_T + [x]_T \subseteq P_T$ for every $x \in \mathcal{M}_v$. In particular, $[1 + x]_T \in P_T$, so $1 + x \in P$, which shows that P is compatible with v.

(b) First observe that $T = \{x \in K \mid v(x) = 0 \text{ and } x \in P\}$. We are going to show that $\text{conv}(T) = P$. By Lemma 4.2.33 (a), it is enough to show that $[x]_T \sim [1]_T$ for every $x \in P$. Take $x \in P$. If $v(x) = 0$, then $x \in T$, so $[x]_T = [1]_T$. In particular, $[x]_T \sim [1]_T$.

Now consider the case of $v(x) > 0$. Let $x - 1 \in P$. Then $[x-1]_T \in ([x]_T - [1]_T) \cap (P_T)_0$. Since $v(x) > 0$, we have that $v(x+1) = \min\{v(x), 0\} = 0$, so $x+1 \in T$. Hence $[1]_T = [1 \cdot (1+x) - x \cdot 1]_T \in ([1]_T - [x]_T) \cap (P_T)_0$, which means that $[x]_T \sim [1]_T$.

If $1 - x \in P$, then $[1-x]_T \in ([1]_T - [x]_T) \cap (P_T)_0$. Since v is not compatible with P, there exists $m \in \mathcal{M}_v \cap P$ such that $m - 1 \in P$. Because v is a rank 1 valuation, there is $n \in \mathbb{N}$ such that $nv(m) > v(x)$. Hence $v(\frac{m^n}{x}) > 0$ and thus $v(1 + \frac{m^n}{x}) = \min\{0, v(\frac{m^n}{x})\} = 0$. What is more, $1 + \frac{m^n}{x} \in P$, so $1 + \frac{m^n}{x} \in T$. We have shown before that also $x + 1 \in T$. Moreover, since $m > 1$, we have $m^n > 1$ as well. Therefore,

$$[m^n - 1]_T = [x \cdot (1 + \frac{m^n}{x}) - 1 \cdot (x+1)]_T \in ([x]_T - [1]_T) \cap (P_T)_0.$$

This shows that $[x]_T \sim [1]_T$.

From Lemma 4.2.8 (d) we know that the incomparability relation is compatible with multiplication. Hence, $[x]_T \sim [1]_T$ if and only if $[x^{-1}]_T \sim [1]_T$. Therefore, the statement is true also for $x \in P$ such that $v(x) < 0$.

We have shown that $\text{conv}(T) = P$, so T is cofinal in K. By Proposition 4.2.34 we obtain that P_T is an archimedean positive cone in K_T. From Corollary 4.2.30 the chain of antichains associated with the positive cone P_T is a short chain, so every two positive elements are incomparable. Therefore, the only case where $[a]_T < [x]_T < [b]_T$ holds is when $a \in -P, x = 0$ and $b \in P$. Clearly $0 \in \mathcal{O}_v$, so the valuation hyperring $\mathcal{O}_v$ is convex. However, by part (a), the valuation v_T is not compatible with P_T. This completes the proof. □

4.3 Real holomorphy hyperrings

The real holomorphy ring of a field K is an important concept in the thoery of ordered fields and real algebraic geometry (for reference, see e.g. [43], [4], Chapter 9 in [28]). In this subsection we are going to define a real holomorphy hyperring and prove that it is a Prüfer hyperring.

Definition 4.3.1. Let F be a real hyperfield and $\mathcal{R}(F)$ be the set of all real valuations of F. The set

$$\text{Hol}(F) := \bigcap_{v \in \mathcal{R}(F)} \mathcal{O}_v$$

is called *the real holomorphy hyperring* of a hyperfield F.

Since, by Lemma 3.1.7, $\mathcal{O}_v$ is a strict subhyperring of F for every valuation v, the set $\text{Hol}(F)$ is a strict subhyperring of F by Lemma 1.3.3.

By Proposition 4.1.7 (a) and Corollary 4.1.20 we have that a valuation v on F is real if and only if there exists a positive cone $P \in \mathcal{X}(F)$ such that $A(P) \subseteq \mathcal{O}_v$. Hence we have

$$\mathrm{Hol}(F) = \bigcap_{P \in \mathcal{X}(K)} A(P).$$

Remark 4.3.2. Observe that $\sum a_i^2 \subseteq \mathrm{Hol}(F)$ if and only if $a_1, ..., a_n \in \mathrm{Hol}(F)$ by Lemma 4.1.21.

Lemma 4.3.3. *Let F be a real hyperfield. Then*

$$\mathrm{Hol}(F) = \{c \in F \mid (I_n \pm c) \cap \sum (F^*)^2 \neq \emptyset \text{ for some } n \in \mathbb{N}\}.$$

Proof. Write $R := \{c \in F \mid (I_n \pm c) \cap \sum (F^*)^2 \neq \emptyset \text{ for some } n \in \mathbb{N}\}$. By Corollary 2.1.17 we have that

$$\sum (F^*)^2 = \bigcap_{P \in \mathcal{X}(K)} P.$$

Observe that

$$\mathrm{Hol}(F) = \{c \in F \mid \forall_{P \in \mathcal{X}(F)} \, \exists_{n \in \mathbb{N}} \, (I_n \pm c) \cap P \neq \emptyset\}$$

and

$$R = \{c \in F \mid \exists_{n \in \mathbb{N}} \, \forall_{P \in \mathcal{X}(F)} \, (I_n \pm c) \cap P \neq \emptyset\}.$$

Hence we immediately have the inclusion $R \subseteq \mathrm{Hol}(F)$.

To prove the reverse inclusion we will use the fact, proven in Theorem 2.1.21, that $\mathcal{X}(F)$ is a Boolean space with a subbasis given by the sets $H(a) = \{P \in \mathcal{X}(F) \mid a \in P\}$. For $n \in \mathbb{N}$, write

$$Y_n(c) := \{P \in \mathcal{X}(F) \mid (I_n \pm c) \cap P \neq \emptyset\}.$$

First we will show that

$$Y_n(c) = \bigcup_{d \in I_n + c} H(d) \; \cap \bigcup_{d \in I_n - c} H(d).$$

Observe that $P \in \bigcup_{d \in I_n + c} H(d)$ if and only if there exists $d \in I_n + c$ such that $P \in H(d)$. The latter is equivalent to $d \in (I_n + c) \cap P$. Hence $Q \in \{P \in \mathcal{X}(F) \mid (I_n + c) \cap P \neq \emptyset\}$ if and only if $Q \in \bigcup_{d \in I_n + c} H(d)$. In other words, for $n \in \mathbb{N}$,

$$\bigcup_{d \in I_n + c} H(d) = \{P \in \mathcal{X}(F) \mid (I_n + c) \cap P \neq \emptyset\}.$$

Similarly one can show that

$$\bigcup_{d \in I_n - c} H(d) = \{P \in \mathcal{X}(F) \mid (I_n - c) \cap P \neq \emptyset\}$$

for $n \in \mathbb{N}$. Moreover, $\bigcup_{d \in I_n + c} H(d)$ and $\bigcup_{d \in I_n - c} H(d)$ are open sets as the unions of open sets. Hence $Y_n(c) = \bigcup_{d \in I_n + c} H(d) \; \cap \bigcup_{d \in I_n - c} H(d)$ and $Y_n(c)$ is open as the intersection of two open

sets. Take now $P \in Y_n(c)$. This means that there exist $d_1, d_2 \in F$ such that $d_1 \in (I_n + c) \cap P$ and $d_2 \in (I_n - c) \cap P$. Then $1 + d_1 \subseteq 1 + I_n + c = I_{n+1} + c$ and $1 + d_1 \subseteq P$, so $(I_{n+1} + c) \cap P \neq \emptyset$. Also $1 + d_2 \subseteq 1 + I_n - c = I_{n+1} + c$ and $1 + d_2 \subseteq P$, so $(I_{n+1} - c) \cap P \neq \emptyset$. Hence $P \in Y_{n+1}(c)$, so the sets $Y_n(c)$ are ordered by inclusion, i.e., $Y_n(c) \subseteq Y_{n+1}(c)$ for every $n \in \mathbb{N}$.

Take $c \in \mathrm{Hol}(F)$. Then for every $P \in \mathcal{X}(F)$ there exists $n \in \mathbb{N}$ such that $(I_n \pm c) \cap P \neq \emptyset$. Hence $\bigcup_{n\in\mathbb{N}} Y_n(c)$ is a cover of $\mathcal{X}(F)$. Since $\mathcal{X}(F)$ is compact, there is a finite subcover, i.e., there is $n_0 \in \mathbb{N}$ such that $Y_{n_0}(c) = \mathcal{X}(F)$. This proves that $\mathrm{Hol}(F) \subseteq R$. □

Lemma 4.3.4. *Let F be a real hyperfield, $\mathrm{Hol}(F)$ the holomorphy hyperring of F and $x, u_i \in F$, $i \in \{1, ..., n\}$. Then:*

(a) $\{x^{-1} \mid x \in 1 + \sum {u_i}^2\} \subseteq \mathrm{Hol}(F)$;

(b) $\{u_i x^{-1} \mid x \in 1 + \sum {u_i}^2\} \subseteq \mathrm{Hol}(F)$;

(c) F is a hyperfield of fractions of $\mathrm{Hol}(F)$.

Proof. Take $x \in 1 + \sum {u_i}^2$. Let v be any real valuation in F. We put $v(u_j) = \min\{v(u_1), ..., v(u_n)\}$ for some $j \in \{1, ..., n\}$. By Remark 4.1.15 we obtain

$$v(x) = \min\{v(1), v(u_1^2), ..., v(u_n^2)\} = 2 \cdot \min\{0, v(u_1), ..., v(u_n)\} = 2 \cdot \min\{0, v(u_j)\}.$$

(a) We consider two cases:

- if $v(u_j) \geq 0$, then $v(x) = 0$. Hence $v(x^{-1}) = 0$, so $x^{-1} \in \mathcal{O}_v$,
- if $v(u_j) < 0$, then $v(x) = 2 \cdot v(u_j) < 0$. Hence $v(x^{-1}) > 0$, so $x^{-1} \in \mathcal{O}_v$.

In both cases $x^{-1} \in \mathcal{O}_v$. This is true for every real valuation v, so $x^{-1} \in \mathrm{Hol}(F)$.

(b) We consider two cases:

- if $v(u_j) \geq 0$, then $v(x) = 0$, so $v(x^{-1}) = 0$. Hence $v(u_i x^{-1}) = v(u_i) + v(x^{-1}) = v(u_i) \geq v(u_j) \geq 0$, so $u_i x^{-1} \in \mathcal{O}_v$.
- if $v(u_j) < 0$, then $v(x) = 2v(u_j) < 0$. So $v(x^{-1}) = -2v(u_j) > 0$. Hence $v(u_i x^{-1}) = v(u_i) + v(x^{-1}) \geq v(u_j) - 2v(u_j) = -v(u_j) > 0$, so $u_i x^{-1} \in \mathcal{O}_v$.

In both cases $u_i x^{-1} \in \mathcal{O}_v$ for every real valuation v of F, so $u_i x^{-1} \in \mathrm{Hol}(F)$.

(c) Choose an element $a \in F$. Take any $t \in 1 + a^2 \subseteq F^*$. Then $a = \frac{a}{1} = \frac{at^{-1}}{t^{-1}}$ by Lemma 1.3.30 (a) and Lemma 1.3.26 and $at^{-1}, t^{-1} \in \mathrm{Hol}(F)$ by part (a) and (b).

□

Theorem 4.3.5. *The real holomorphy hyperring $\mathrm{Hol}(F)$ of a real hyperfield F is a Prüfer hyperring.*

Proof. Write $R = \mathrm{Hol}(F)$ and let $\mathfrak{p}$ be any prime hyperideal in R. We are going to show that $R_{\mathfrak{p}}$ is a valuation hyperring. Let us start with showing that for every $a \in F^*$ we have $a^2 \in R_{\mathfrak{p}}$ or $a^{-2} \in R_{\mathfrak{p}}$. For $y \in F$ let us write

$$S_y := \{x^{-1} \in F \mid x \in 1 + y^2\}.$$

From Lemma 4.3.4 (a) and Lemma 1.3.30 (a) we have $S_y \subseteq R \subseteq R_{\mathfrak{p}}$ for every $y \in F$.

Suppose that there exists $a \in F^*$ such that $a^2 \notin R_{\mathfrak{p}}$ and $a^{-2} \notin R_{\mathfrak{p}}$. If there exists $x \in (1+a^2) \cap R_{\mathfrak{p}}$, then from the reversibility axiom and from the fact that $R_{\mathfrak{p}}$ is additively closed, we have $a^2 \in x - 1 \subseteq R_{\mathfrak{p}}$, which is a contradiction. Hence $(1 + a^2) \cap R_{\mathfrak{p}} = \emptyset$. Similarly, we obtain $(1 + a^{-2}) \cap R_{\mathfrak{p}} = \emptyset$. From what we have just shown, we have that if $x \in S_a \subseteq R_{\mathfrak{p}}$, then $x^{-1} \notin R_{\mathfrak{p}}$. So from Lemma 3.1.8,

$$S_a \subseteq R_{\mathfrak{p}} \setminus R_{\mathfrak{p}}^{\times} = \mathcal{M}_{R_{\mathfrak{p}}}.$$

Similarly, $S_{a^{-1}} \subseteq \mathcal{M}_{R_{\mathfrak{p}}}$. Observe that

$$1 + a^{-2} = a^2 a^{-2} + a^{-2} = a^{-2}(1 + a^2).$$

Hence $S_{a^{-1}} = a^2 S_a$. We have

$$1 \in (1 + a^2)S_a \subseteq S_a + a^2 S_a = S_a + S_{a^{-1}} \subseteq \mathcal{M}_{R_{\mathfrak{p}}},$$

where for the first inclusion we used Lemma 1.2.3. We obtain a contradiction, so for every $a \in F^*$ we have $a^2 \in R_{\mathfrak{p}}$ or $a^{-2} \in R_{\mathfrak{p}}$.

Now we are going to show that if $a^2 \in R_{\mathfrak{p}}$, then $a \in R_{\mathfrak{p}}$ for every $a \in F$. Take $a^2 \in R_{\mathfrak{p}}$. Then $1 + a^2 \subseteq R_{\mathfrak{p}}$, because $R_{\mathfrak{p}}$ is additively closed. Moreover, we have already observed that $S_a \subseteq R_{\mathfrak{p}}$, so $1 + a^2 \subseteq R_{\mathfrak{p}}^{\times}$. We have that $aS_a \subseteq R \subseteq R_{\mathfrak{p}}$, where the first inclusion follows from Lemma 4.3.4 (b) and the second inclusion from Lemma 1.3.30 (a). Take $x \in S_a$. Then

$$a = ax \cdot x^{-1} \in aS_a \cdot (1 + a^2) \subseteq R_{\mathfrak{p}}.$$

Similarly we show that if $a^{-2} \in R_{\mathfrak{p}}$, then $a^{-1} \in R_{\mathfrak{p}}$. Since also $0 \in R_{\mathfrak{p}}$, we conclude that for every $a \in F$ we have $a \in R_{\mathfrak{p}}$ or $a^{-1} \in R_{\mathfrak{p}}$, so $R_{\mathfrak{p}}$ is a valuation hyperring. This means that $\mathrm{Hol}(F)$ is a Prüfer hyperring. □

4.4 Real places in hyperfields

Let K, L be fields and ∞ a formal symbol. We extend operations on L into $L \cup \{\infty\}$ by setting

$$\infty \cdot \infty = \infty, \qquad \frac{1}{0} = \infty, \qquad \frac{1}{\infty} = 0, \qquad x \pm \infty = \infty \text{ for } x \in L, \qquad x \cdot \infty = \infty \text{ for } 0 \neq x \in L.$$

The expressions: $\infty \pm \infty, 0 \cdot \infty, \frac{0}{0}, \frac{\infty}{\infty}$ are not defined. In the classical theory, a place on a field K with values in L is usually defined as follows:

Definition 4.4.1 (Definition 2.8.1, [22]). The map $\lambda : K \to L \cup \{\infty\}$ is called a *place* on K with values in L if

$$\lambda(1) = 1, \qquad \lambda(x+y) = \lambda(x) + \lambda(y), \qquad \lambda(x \cdot y) = \lambda(x) \cdot \lambda(y)$$

for all $x, y \in K$ whenever the right-hand sides are defined.

If L is real, then the place λ is called a *real place*, and if $L = \mathbb{R}$, then λ is called an $\mathbb{R}$-*place*. Since in hyperfield theory it is difficult to speak about the analogues of real numbers, we define a real place of a hyperfield to be a pair of a real valuation hyperring and a positive cone in the residue hyperfield. In case the hyperfield is a field, this definition is equivalent to the one given above (see Proposition 3.1 in [13]) .

Definition 4.4.2. Let F be a real hyperfield. Let $\mathcal{O}$ be a valuation hyperring on F such that its residue hyperfield $\overline{F} := \mathcal{O}/\mathcal{M}$ is real and admits an archimedean positive cone $\overline{P}$. Then the pair $(\mathcal{O}, \overline{P})$ is called a *real place*. We denote by $M(F)$ the set of all real places on F.

Proposition 4.4.3. *Let F be a real hyperfield. Consider the map*

$$\begin{aligned} \Lambda_F : \mathcal{X}(F) &\to M(F) \\ P &\mapsto (A(P), \overline{P}), \end{aligned}$$

where $\overline{P} = \overline{P \cap A(P)^\times}$ is a positive cone in the residue hyperfield $A(P)/I(P)$ induced by P. Then λ_F is surjective.

Proof. Take $(\mathcal{O}, Q) \in M(F)$. Then, by Proposition 3.1.11, there exists a valuation v such that $\mathcal{O} = \mathcal{O}_v$. Moreover, v is real, because its residue hyperfield $\overline{F_v} = \mathcal{O}_v/\mathcal{M}_v$ admits a positve cone Q. By Proposition 4.1.19 there exists a positive cone $P \in \mathcal{X}(F)$ compatible with v such that $\overline{P} = Q$. Hence $Q = \overline{P \cap \mathcal{O}_v^\times}$. Since Q is an archimedean positive cone, from Lemma 4.1.18 we have that v is the natural valuation associated with P. Hence $(\mathcal{O}, Q) = (A(P), \overline{P}) = \Lambda_F(P)$. □

Since Λ_F is surjective, we can endow $M(F)$ with the quotient topology induced by $\mathcal{X}(F)$, i.e., for the open sets in $M(K)$ we take such $U \in M(K)$ that $\Lambda_F^{-1}(U)$ is open in $\mathcal{X}(F)$. Hence Λ_F is continuous. The image of a compact topological space under a continuous map is compact (see e.g. Theorem 7.8 in [21]). Hence $M(F)$ is compact.

Lemma 4.4.4. *Let F be a real hyperfield. The set $U(a) := \{(A(P), \overline{P}) \in M(K) \mid \overline{a} \in \overline{P}\}$ for $a \in F$ is open in $M(F)$. Moreover, for every $a \in F$, the sets $U(a)$ and $U(-a)$ are disjoint.*

Proof. First recall that

$$A(P)^\times = \{a \in F \mid (I_{n_1} \pm a) \cap P \neq \emptyset \text{ and } (I_{n_2} \pm a^{-1}) \cap P \neq \emptyset \text{ for some } n_1, n_2 \in \mathbb{N}\}.$$

If $(I_n \pm a) \cap P \neq \emptyset$, then also $(I_{n+1} \pm a) \cap P \neq \emptyset$. Hence, for $n = \max\{n_1, n_2\}$ we have

$$A(P)^\times = \{a \in F \mid (I_n \pm a) \cap P \neq \emptyset \text{ and } (I_n \pm a^{-1}) \cap P \neq \emptyset\}.$$

Assume that $a \in P$. Then $I_n + a \subseteq P$ and $I_n + a^{-1} \subseteq P$ for every $n \in \mathbb{N}$. So we can write

$$P \cap A(P)^\times = \{a \in P \mid (I_n - a) \cap P \neq \emptyset \text{ and } (I_n - a^{-1}) \cap P \neq \emptyset \text{ for some } n \in \mathbb{N}\}.$$

Recall that the sets $H(a) = \{P \in \mathcal{X}(F) \mid a \in P\}$ form a subbasis of the topological space $\mathcal{X}(F)$ (see Theorem 2.1.21). Consider the preimage of the set $U(a)$ under Λ_F:

$$\begin{aligned}\Lambda_F^{-1}(U(a)) &= \{P \in \mathcal{X}(F) \mid \overline{a} \in \overline{P}\} = \{P \in \mathcal{X}(F) \mid a \in P \cap A(P)^\times\} \\ &= H(a) \cap \bigcup_{n \in \mathbb{N}} \Big(\bigcup_{d \in I_n - a} H(d) \cap \bigcup_{d \in I_n - a^{-1}} H(d) \Big).\end{aligned}$$

The second and the third equalities require more explanation. For the second equality we use Proposition 4.1.7 (b) and the fact that if $\overline{a} = \overline{b}$ and $a \in P \cap A(P)^\times$, then $b \in P \cap A(P)^\times$, which is a part of the proof of that proposition. One can prove the third equality in a similar way as it is done in the proof of Lemma 4.3.4. Since the sets $H(a)$ form a subbasis of a topological space $\mathcal{X}(F)$, they are open in $\mathcal{X}(F)$. Hence $\Lambda_F^{-1}(U(a))$ is open in $\mathcal{X}(F)$ as a finite intersection of open sets. This means that $U(a)$ is open in the quotient topology on $M(F)$.

Suppose that $U(a)$ and $U(-a)$ are not disjoint for some $a \in F$, i.e., there exists $(A(P), \overline{P}) \in U(a) \cap U(-a)$. But then $\overline{a} \in \overline{P}$ and $\overline{-a} \in \overline{P}$, which is a contradiction. □

The space of real places on a field is a Hausdorff space (see e.g. Corollary 9.9 in [28]). The same is true for the space of places on a hyperfield, which we will see in the next theorem.

Theorem 4.4.5. *Let F be a real hyperfield and consider the space $M(F)$ of real places on F. The space $M(F)$ is a Hausdorff space.*

Proof. Take $(\mathcal{O}_1, \overline{P}), (\mathcal{O}_2, \overline{Q}) \in M(F)$. From Proposition 3.1.11 there exist valuations v and w on F such that $\mathcal{O}_1 = \mathcal{O}_v$ and $\mathcal{O}_2 = \mathcal{O}_w$. From Proposition 4.4.3 we know that there are $P_1, P_2 \in \mathcal{X}(F)$ such that v is the natural valuation associated with P_1 and w is the natural valuation associated with P_2, i.e., $\mathcal{O}_v = A(P_1)$ and $\mathcal{O}_2 = A(P_2)$. Also $\overline{P} = \overline{P_1}$ and $\overline{Q} = \overline{P_2}$. Assume that $(A(P_1), \overline{P_1}) \neq (A(P_2), \overline{P_2})$. Then either $A(P_1) \neq A(P_2)$ or $A(P_1) = A(P_2)$, but $\overline{P_1} \neq \overline{P_2}$. In the latter case, there is $a \in F$ such that $\overline{a} \in \overline{P_1}$ and $-\overline{a} \in \overline{P_2}$. Hence $(A(P_1), \overline{P_1}) \in U(a)$ and $(A(P_2), \overline{P_2}) \in U(-a)$.

Consider now the case when $A(P_1) \neq A(P_2)$. This means that there is $a \in F$ such that $a \in A(P_1)$ and $a \notin A(P_2)$. Without loss of generality, assume that $a \in P_1$. Consider the set $1 + a$ and take any $x \in F$ such that $x^{-1} \in 1 + a$. Since $v(a) \geq 0$, we obtain that $v(x) = 0$. Since $w(a) < 0$, we have that $w(x) > 0$. Hence, $x \in A(P_1)^\times$ and $x \in I(P_2)$.

Since $x \notin I(P_1)$, there is $n \in \mathbb{N}$ such that $(1 - I_n \cdot x) \cap -P_1 \neq \emptyset$. Take $y \in (1 - I_n \cdot x) \cap -P_1$. Then $v(y) \geq 0$ by axiom (V3) from Definition 3.1.1 and Corollary 4.1.14. Moreover, $w(y) = 0$ by Lemma 3.1.2 (d) and Corollary 4.1.14. Once again we consider two cases: $v(y) = 0$ or $v(y) > 0$.

Let $v(y) = 0$. Then $y \in A(P_1)^\times$. Moreover, $y \in -P_1$, so $\overline{y} \in -\overline{P_1}$. Since $w(y) = 0$, we have $y \in A(P_2)^\times$. Because $x \in I(P_2)$, we obtain that $I_n \cdot x \subseteq I(P_2)$. Then

$$y \in 1 - I_n \cdot x \subseteq 1 + I(P_2) \subseteq P_2,$$

where for the last inclusion we used the fact that w is compatible with P_2 and Proposition 4.1.7 (c). So $y \in P_2$, which implies that $\overline{y} \in \overline{P_2}$. We conclude that $(A(P_1), \overline{P_1}) \in U(-y)$ and $(A(P_2), \overline{P_2}) \in U(y)$.

Let now $v(y) > 0$. Then $v(t) = 0$ and $w(t) = 0$ for every $t \in y - x$. Hence $t \in A(P_1)^\times \cap A(P_2)^\times$. Observe that

$$t \in y - x \subseteq 1 - I_n \cdot x - x = 1 - I_{n+1} \cdot x \subseteq 1 + I(P_2) \subseteq P_2.$$

The last inclusion follows from the fact that w is compatible with P_2 and from Proposition 4.1.7 (c). Note that $x \in P_1$, because we have choosen $x^{-1} \in 1 + a \subseteq P_1$. Hence, $y - x \subseteq -P_1$, so $t \in -P_1$. Finally, we conclude that $(A(P_1), \overline{P_1}) \in U(-t)$ and $(A(P_2), \overline{P_2}) \in U(t)$.

In every case, we obtained that there are sets $U(c), U(-c)$ with $c \in \{a, y, t\} \subseteq A(P_1)^\times \cap A(P_2)^\times$ which separate two different elements from $M(F)$. These sets are open and disjoint by Lemma 4.4.4. $\square$

Bibliography

[1] E. Artin, *Über die Zerlegung definiter Funktionen in Quadrate*, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, 5(1), (1927), 100-115

[2] E. Artin, O. Schreier, *Algebraische Konstruktion reeller Körper*, Vol. 5, Springer, 1927, 85-99

[3] M. Baker, T. Jin, *On the structure of hyperfields obtained as quotients of fields*, Proceedings of the American Mathematical Society 149(1), (2021), 63-70

[4] E. Becker, D. Gondard, *Notes on the space of real places of a formally real field*, Real Analytic and Algebraic Geometry, Trento Conference, (1992), 21-46

[5] V. Bergelson, D. B. Shapiro, *Multiplicative subgroups of finite index in a ring*, Proceedings of the American Mathematical Society 116(4), (1992), 885-896

[6] J. Bochnak, M. Coste, M.-F. Roy, *Real algebraic geometry*, Vol. 36, Springer Science Business Media, 2013

[7] N. Bowler, T. Su, *Classification of doubly distributive skew hyperfields and stringent hypergroups*, Journal of Algebra 574, (2021), 669-698

[8] P. Corsini, *Hypergroupes régulières et hypermodules*, Annali dell'Università di Ferrara 20, (1975), 121-135

[9] P. Corsini, V. Leoreanu, *Applications of hyperstructure theory*, Vol. 5, Springer Science and Business Media, 2013

[10] B. Davvaz, V. Leoreanu-Fotea, *Hyperring theory and applications*, Vol. 347, International Academic Press, USA, 2007

[11] B. Davvaz, A. Salasi, *A realization of hyperrings*, Communications in Algebra 34(12), (2006), 4389-4400

[12] I. Efrat, *Valuations, orderings, and Milnor K-theory*, Vol 124, American Mathematical Society, 2006

[13] I. Efrat, K. Osiak, *Topological spaces as spaces of R-places*, Journal of Pure and Applied Algebra 215(5), (2011), 839-846

[14] O. Endler, *Valuation theory*, Vol. 5, Berlin, Springer, 1972

[15] M. Fontana, J. Huckaba, I. Papick, *Prüfer domains*, CRC Press, 1996

[16] P. Gładki, M. Marshall, *Orderings and signatures of higher level on multirings and hyperfields*, Journal of K-Theory 10(3), (2012), 489-518

[17] H. Hahn, *Über die nichtarchimedischen Größensysteme*, Hans Hahn Gesammelte Abhandlungen Band 1 / Hans Hahn Collected Works Volume 1, Springer, Vienna, 1995, 445-499

[18] J. Jun, *Algebraic geometry over hyperrings*, Advances in Mathematics 323, (2018), 142-192

[19] J. Jun, *Algebraic geometry over semi-structures and hyper-structures of characteristic one*, PhD Thesis, Johns Hopkins University, 2015

[20] D. E. Kędzierski, A. Linzi, H. Stojałowska, *Characteristic, C-characteristic and positive cones in hyperfields*, Mathematics, 11(3), (2023), 779

[21] C. Kosniowski, *A first course in algebraic topology*, Cambridge University Press, 1980

[22] M. Knebusch, C. Scheiderer, *Real Algebra: A First Course*, Springer Nature, 2022

[23] M. Krasner, *A class of hyperrings and hyperfields*, International Journal of Mathematics and Mathematical Sciences 6, (1983), 307-311

[24] M. Krasner, *Approximation des corps valués complets de caractéristique p= 0 par ceux de caractéristique 0*, Colloque d'algèbre supérieure, tenu à Bruxelles du 19, (1956), 129-206

[25] F.-V. Kuhlmann, *Valuation theory*, preprint, https://www.fvkuhlmann.de/Fvkbook.htm

[26] K. Kuhlmann, A. Linzi, H. Stojałowska, *Orderings and valuations in hyperfields*, Journal of Algebra 611, (2022), 399-421

[27] M. Kula, A. Sładek, *Teoria ciał uporządkowanych*, Wydawnictwo Uniwersytetu Śląskiego, Katowice, 2013

[28] T.-Y. Lam, *Orderings, valuations and quadratic forms*, Vol.52, American Mathematical Society, 1983

[29] J. Lee, *Hyperfields, truncated DVRs, and valued fields*, Journal of Number Theory 212, (2020) 40-71

[30] A. Linzi, H. Stojałowska, *Hypervaluations on hyperfields and ordered canonical hypergroups*, arXiv:2009.08954, to appear in Iranian Journal of Mathematical Sciences and Informatics

[31] F. Marty, *Sur une generalization de la notion de groups*, 8th Congress Math. Scandinaves, Stockholm, (1934)

[32] C. Massouros, *Methods of constructing hyperfields*, International Journal of Mathematics and Mathematical Sciences 8, (1985), 725-728

[33] C. Massouros, G. Massouros, *An overview of the foundations of the hypergroup theory*, Mathematics 9(9), (2021), 1014

[34] M. Marshall, *Real reduced multirings and multifields*, Journal of Pure and Applied Algebra, 205(2), (2006), 452-468

[35] M. Marshall, *Spaces of orderings and abstract real spectra*, Lecture Notes in Mathematics vol. 1636, Springer, Berlin Heidelberg, 1996

[36] J. Maxwell, B. Smith, *Convex geometry over ordered hyperfields*, arXiv:2301.12760, (2023)

[37] KH. Mirdar Harijani, S. Anvariyeh, *A hypervaluation of a hyperfield onto a totally ordered canonical hypergroup*, Studia Scientiarum Mathematicarum Hungarica, 52(1), (2015), 87-101

[38] J. Mittas, *Hypergroupes canoniques*, Mathematica Balkanica 2, (1972), 165-179

[39] J. Mittas, *Sur les hyperanneaux et les hypercorps*, Mathematica Balkanica 3, (1973), 368-382

[40] A. Nakassis, *Recent results in hyperring and hyperfield theory*, International Journal of Mathematics and Mathematical Sciences 11, (1988), 209-220

[41] A. Prestel, *Lectures on formally real fields*, Vol. 1093, Springer, 2007

[42] R. Rota, *Sugli iperanelli moltiplicativi*, Rend. Di Mat., Series 7(4), 2, (1982)

[43] H. W. Schülting, *Real holomorphy rings in real algegraic geometry*, Lecture Notes in Mathematics 959, (1982), 433-442

[44] L. A. Steen, J. A. Seebach, *Counterexamples in topology*, Vol. 18, New York, Springer, 1978

[45] G. Turnwald, *Multiplicative subgroups of finite index in a division ring*, Proceedings of the American Mathematical Society, 120(2), (1994) 377-381

[46] O. Viro, *Hyperfields for tropical geometry I. Hyperfields and dequantization*, arXiv:1006.3034, (2010)

Uniwersytet Szczeciński

Instytut Matematyki

mgr Hanna Stojałowska

Tytuł rozprawy doktorskiej: **Real hyperfields**

Promotor: **prof. dr hab. Franz-Viktor Kuhlmann**

Streszczenie rozprawy doktorskiej

Hiperciała są uogólnieniem ciał, w których dodawanie jest wielowartościowe, a mnożenie jednowartościowe. Zostały wprowadzone przez Krasnera w 1956 roku. W 2006 roku Marshall wprowadził pojęcie hiperciała rzeczywistego. Mówimy, że hiperciało jest rzeczywiste, jeśli ma co najmniej jeden porządek. W niniejszej rozprawie rozwijamy teorię hiperciał rzeczywistych. Wprowadzamy pojęcie porządku zgodnego z waluacją w hiperciałach i udowadniamy analog twierdzenia Baera-Krulla dla hiperciał. Ponadto uogólniamy kilka aspektów teorii hiperciał rzeczywistych, pochodzących z klasycznej teorii ciał uporządkowanych. Na przykład, wprowadzamy pojęcie rzeczywistego hiperpierścienia holomorficznego i pokazujemy, że jest to hiperpierścień Prufera (który również definiujemy w tej rozprawie). Co więcej, definiujemy punkty rzeczywiste na hiperciele i pokazujemy, że przestrzeń topologiczna punktów rzeczywistych na hiperciele ma własności analogiczne do tych z klasycznej teorii ciał uporządkowanych. Jednakże, niektóre fakty znane z klasycznej teorii ciał rzeczywistych nie mogą zostać uogólnione na teorię hiperciał. W niniejszej rozprawie podajemy przykłady takich faktów.

W 1983 roku Krasner przedstawił konstrukcję hiperciała, które jest ilorazem ciała przez multiplikatywną podgrupę tego ciała. Takie hiperciała nazywamy hiperciałami faktoryzowanymi. W niniejszej rozprawie badamy rzeczywiste hiperciała faktoryzowane. Między innymi, badamy relację ostrego, częściowego porządku związanego z porządkiem hiperciała. W szczególności skupiamy się na elementach, które nie są porównywalne. Pokazujemy, że każde rzeczywiste hiperciało faktoryzowane jest sumą mnogościową łańcucha antyłańcuchów i określamy liczbę elementów w każdym antyłańcuchu. Ponadto pokazujemy związek między porządkami w ciele i porządkami rzeczywistego hiperciała faktoryzowanego, powstałego z tego ciała.

……

słowa kluczowe w języku polskim (odpowiedniki słów kluczowych w języku angielskim).

Hiperciało (hyperfield), hiperstruktury (hyperstructures), waluacja (valuation), porządek (positive cone)

University of Szczecin

Institute of Mathematics

mgr Hanna Stojałowska

Title of the dissertation: **Real hyperfields**

Supervisor: **prof. dr hab. Franz-Viktor Kuhlmann**

Summary of the dissertation

Hyperfields are a generalization of fields in which addition is multivalued and multiplication is single-valued. They were introduced by Krasner in 1956. In 2006 Marshall introduced the concept of a real hyperfield. We say that a hyperfield is real if it admits at least one positive cone. In this thesis, we develop the theory of real hyperfields. We introduce the notion of a positive cone compatible with a valuation in hyperfields and prove an analogue of the Baer-Krull theorem for hyperfields. In addition, we generalize several aspects of the theory of real hyperfields from the classical theory of ordered fields. For example, we introduce the notion of a real holomorphy hyperring and show that it is a Prufer hyperring (which we also define in this thesis). Moreover, we define real places on hyperfields and show that the topological space of real places on hyperfields has properties analogous to those of the classical theory of ordered fields. However, some facts from the classical theory of real fields cannot be generalized to the theory of hyperfields. In this thesis, we give examples of such facts.

In 1983, Krasner presented the construction of a hyperfield, which is the quotient of a field by a multiplicative subgroup of that field. We call these hyperfields factor hyperfields. In this thesis, we study real factor hyperfields. For instance, we study the relation of a strict partial order associated with the positive cone of a hyperfield. In particular, we focus on elements that are not comparable. We show that each real factor hyperfield is a union of a chain of antichains and we determine the number of elements in each antichain. In addition, we show the relation between the positive cones in a field and the positive cones of a real factor hyperfield arising from that field.

……

słowa kluczowe w języku polskim (odpowiedniki słów kluczowych w języku angielskim).

Hiperciało (hyperfield), hiperstruktury (hyperstructures), waluacja (valuation), porządek (positive cone)